Involution and Constrained Dynamics I: The Dirac Approach

Werner M. Seiler[⋆], *Robin W. Tucker*

School of Physics and Materials
Lancaster University
Bailrigg, LA1 4YB, UK
Email: W.Seiler@lancaster.ac.uk, R.W.Tucker@lancaster.ac.uk

Abstract

We study the theory of systems with constraints from the point of view of the formal theory of partial differential equations. For finite-dimensional systems we show that the Dirac algorithm completes the equations of motion to an involutive system. We discuss the implications of this identification for field theories and argue that the involution analysis is more general and flexible than the Dirac approach. We also derive intrinsic expressions for the number of degrees of freedom.

1 Introduction

Constrained dynamics represents a cornerstone of theoretical physics, as every relativistic theory and every theory with gauge symmetries necessarily possesses constraints. Thus it is not very surprising that many methods for dealing with such systems have been developed (see e.g. [12, 15, 19, 33, 34]). The purpose of this and the following articles in this series is to present an alternative ansatz based on the modern theory of differential equations and especially on the concept of involution [23, 28].

The classical Hamiltonian treatment of systems with constraints was developed by Dirac [4, 5, 6]. We will show that in the case of finite-dimensional systems his algorithm corresponds to rendering the equations of motion involutive. In the language of exterior differential systems this was already noted by Hartley *et al.* [14]. But we will also show that this connection does no longer hold for field theories. Here it might happen that the Dirac analysis alone is not sufficient to obtain all constraints.

This identification appears natural, as the basic idea behind the Dirac algorithm is to check whether or not the equations of motions are consistent. But the notion of involution represents essentially a mathematical formulation of this problem. We believe that this approach has conceptual and practical advantages,

⋆ On leave from Institut für Algorithmen und Kognitive Systeme, Universität Karlsruhe, 76128 Karlsruhe, Germany

especially in the case of field theories. The theory of involution is well understood for arbitrary systems. The Cartan-Kuranishi Theorem [18, 23] yields a general procedure to complete any system of partial differential equations to an involutive one.

In contrast one can find in physics many different approaches depending on whether one deals with a system in Lagrangian or Hamiltonian formulation, or whether the Lagrangian contains higher order derivatives or is linear in the velocities. Each case is handled individually in the literature. A closer analysis shows, however, that almost all proposed methods represent nothing else than special cases of the general completion procedure stemming from the Cartan-Kuranishi Theorem[2].

We study in this article the standard situation of a system described by a Lagrangian depending on the generalized coordinates and the velocities as well as Lagrangians containing higher order derivatives. The special case of a Lagrangian being linear in the velocities and the so-called symplectic formalism of Faddeev and Jackiw [8] will be considered in the next article in this series. As concrete examples we consider among others the rigid rotator and Podolsky's generalized electrodynamics.

Special emphasis is put on the problem of determining the number of degrees of freedom. We propose a new intrinsic definition for field theories using the Cartan characters of the field equations. It can also handle systems described in characteristic coordinates like light-cone coordinates. The classical approach based on a distinction into first and second class constraints fails in such a situation [32], as seemingly too many constraints occur.

The article is organized as follows: The next two sections serve as a brief introduction into the formal theory of differential equation. They define the notion of involution and show how one completes an arbitrary system to an involutive one. Section 4 reviews the classical Dirac approach. In Section 5 we show its relation to the formal analysis of the Hamiltonian equations of motion for finite-dimensional systems. A detailed example is considered in Section 7. Section 8 contains an example of a field theory where the Dirac algorithm alone does not suffice to exhibit the full constraint structure. The problem of counting degrees of freedom is tackled in Sections 6 and 10 for the finite and infinite-dimensional case, respectively. Finally, before some conclusions are given, we consider in Section 9 Lagrangians depending on higher order derivatives.

2 Involution

Formal theory uses a geometric approach to differential equations based on the jet bundle formalism. It is beyond the scope of this paper to give a detailed introduction into the underlying theory. The interested reader is referred to the literature [23, 26, 28]. Here we are concerned with two topics: the definition of an

[2] This holds even for the approach of Lusanna [19] via the Noether theorem, as it is based on the eigenvectors of the Hessian. But these correspond exactly to the linear combinations of the equations of motion which yield the integrability conditions.

involutive system and how to compute the arbitrariness of the general solution of such a system.

We will always work in a local coordinate system, although the whole theory can be expressed in a coordinate free way. Let X denote the space of the independent variables $x_1, \dots, x_n$ and let the dependent variables $u^1, \dots, u^m$ be fiber coordinates for the bundle $\mathcal{E}$ over the base space X. Derivatives are written in multi-index notation $p^\alpha_\mu = \partial^{|\mu|} u^\alpha / \partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}$ where $|\mu| = \mu_1 + \cdots + \mu_n$ is the length of the multi-index $\mu = [\mu_1, \dots, \mu_n]$. Adding the derivatives p^α_μ up to order q defines a local coordinate system for the q-th order jet bundle $J_q\mathcal{E}$. A system of differential equation $\mathcal{R}_q$ of order q can be described locally by

$$\mathcal{R}_q \;:\; \left\{ \Phi^\tau \left(x_i, u^\alpha, p^\alpha_\mu\right) = 0\,, \quad \tau = 1, \dots, p\,;\; |\mu| \le q\,. \right. \tag{1}$$

Geometrically, this represents a fibered submanifold of $J_q\mathcal{E}$.

At least some of the ideas behind the concept of involution can be understood best by considering the order by order construction of a formal power series solution. For this purpose, we introduce the symbol $\mathcal{M}_q$ of a differential equation $\mathcal{R}_q$.

Definition 1. The *symbol* $\mathcal{M}_q$ of the system (1) is the solution space of the following linear system of (algebraic!) equations in the unknowns v^α_μ

$$\mathcal{M}_q \;:\; \left\{ \sum_{\alpha, |\mu| = q} \left(\frac{\partial \Phi^\tau}{\partial p^\alpha_\mu} \right) v^\alpha_\mu = 0\,. \right. \tag{2}$$

(By abuse of language, we will refer to both the linear system and its solution space as the symbol).

The placeholders v^α_μ are coordinates of a finite-dimensional vector space, i.e. we introduce one coordinate for each derivative of order q. Definition 1 is most easily understood by considering a quasi-linear system, i.e. a system linear in the derivatives p^α_μ with $|\mu| = q$. For such a system the symbol is simply obtained by taking only the linear highest order part and substituting v^α_μ for p^α_μ.

We make a power series ansatz for the general solution of the differential equation $\mathcal{R}_q$ by expanding around some point x^0

$$u^\alpha(x) = \sum_{|\mu|=0}^{\infty} \frac{a^\alpha_\mu}{\mu!} (x - x^0)^\mu \tag{3}$$

and substitute this ansatz into the equations (1) evaluating at x^0. This yields a system of algebraic equations for the Taylor coefficients a^α_μ up to order q.

The remaining coefficients can be computed by linear algebra only. For the coefficients of order $q + r$ we use the *prolonged* systems $\mathcal{R}_{q+r}$ which are obtained by differentiating each equation in $\mathcal{R}_q$ r times totally with respect to all independent variables. They are all quasi-linear. If we substitute again the power series ansatz into the prolonged system $\mathcal{R}_{q+r}$ and evaluate at x^0, we get an inhomogeneous linear system for the coefficients of order $q+r$. Its homogeneous part is determined by the prolonged symbol $\mathcal{M}_{q+r}$, i.e. the symbol of $\mathcal{R}_{q+r}$.

The Taylor coefficients a^α_μ of lower order appear in the matrix and in the right hand side of this linear system. Thus we are able to express the coefficients of order $q+r$ through the coefficients of lower order. This is the precise meaning of constructing a power series order by order.

This construction will fail, if non-trivial integrability conditions occur, i.e. equations of order $q+r$ which are functionally independent of the equations contained in the prolonged system $\mathcal{R}_{q+r}$ and which are satisfied by every solution of the system. Such equations arise usually by cross-differentiating and are detected only in some higher prolongation. They pose additional conditions on the coefficients of order $q+r$. Hence they must all be known to pursue the above described procedure. We call a system which contains all its integrability conditions a *formally integrable* system.

For formally integrable systems it is thus possible to construct order by order a formal power series solution. The arbitrariness of the general solution is reflected by the dimensions of the prolonged symbols, because at each order $\dim \mathcal{M}_{q+r}$ coefficients are not determined by the differential equations but can be chosen freely [29]. Formal integrability does, however, not suffice to determine these dimensions in advance without explicitly constructing the prolonged symbols. This leads to the concept of involution.

We introduce the *class* of a multi-index $\mu = [\mu_1, \dots, \mu_n]$. It is the smallest k for which μ_k is different from zero. If we consider the symbol (2) as a matrix, then its columns are labeled by the coordinates v^α_μ. We order them by class, i.e. we always take a column with a multi-index of higher class left of one with lower class. Then we compute a row echelon form.

In this solved form the symbol is especially easy to analyze. Since we only need linear operations to obtain it, we can always perform the same operations with the full system $\mathcal{R}_q$ and thus assume that (2) yields the symbol directly in solved form. We denote the number of rows where the leading entry or pivot is of class k by $\beta^{(k)}_q$ and we associate with each such row its *multiplicative variables* $x_1, \dots, x_k$.

It is important to note that if we prolong each equation only with respect to its multiplicative variables, we obtain independent equations, because each equation will have a different leading term. The question is, whether prolongation with respect to the non-multiplicative variables leads to additional independent equations. If not we call the symbol involutive.

Definition 2. The symbol $\mathcal{M}_q$ is called *involutive*, if

$$\operatorname{rank} \mathcal{M}_{q+1} = \sum_{k=1}^{n} k\beta^{(k)}_q \,. \tag{4}$$

The system $\mathcal{R}_q$ is called *involutive*, if it is formally integrable and its symbol is involutive.

The above definition of the $\beta_q^{(k)}$ is obviously coordinate dependent. Thus it seems, as if the involution of a symbol depends on the chosen coordinate system, too. One can, however, show that almost every coordinate system leads to the same values for the $\beta_q^{(k)}$. These values are characterized by the property that all the sums $\sum_{i=k}^{n} \beta_q^{(i)}$, $k = 1, \dots, n$, are maximal.[3] A coordinate system which leads to these values is called *δ-regular*. Definition 2 assumes that the $\beta_q^{(k)}$ are computed in such a coordinate system. Besides there exist alternative methods to obtain the correct values intrinsically [23, 31, 28]. We will return to this point in Section 10.

The prolongation of an involutive symbol is again involutive. Since prolonging an equation with respect to one of its multiplicative variables x_i yields an equation of class i, we get $\beta_{q+1}^{(i)} = \sum_{k=i}^{n} \beta_q^{(k)}$. Inductive use of this relation leads to

$$\beta_{q+r}^{(k)} = \sum_{i=k}^{n} \binom{r+i-k-1}{r-1} \beta_q^{(i)} \tag{5}$$

and together with Definition 2 to

$$\operatorname{rank} \mathcal{M}_{q+r} = \sum_{k=1}^{n} \binom{r+k-1}{r} \beta_q^{(k)} . \tag{6}$$

Besides the possibility to predict the number of arbitrary Taylor coefficients at any order, involutive systems have another advantage compared with formally integrable ones. There exists an easily applicable criterion to check whether or not a system is involutive. The problem of the definition of formal integrability is that one has to prove that a system does not generate non-trivial integrability conditions at any prolongation order, i.e. one must check an infinite number of conditions. This can, however, be done in a finite manner for systems with an involutive symbol.

Theorem 3. *Let $\mathcal{R}_q$ be a q-th order differential equation with an involutive symbol $\mathcal{M}_q$. If no integrability conditions arise during the prolongation of $\mathcal{R}_q$ to $\mathcal{R}_{q+1}$, then $\mathcal{R}_q$ is involutive.*

3 Completion to Involution and Arbitrariness

Since we have seen that involutive systems have many advantages, the question naturally arises whether they form only a very special class of systems and what to do with a non-involutive system. The interesting answer is given by the Cartan-Kuranishi Theorem [18, 23, 28].

Theorem 4. *Any system $\mathcal{R}_q$ can be completed to an equivalent involutive one by a finite number of prolongations and projections (i.e. addition of integrability conditions).*

[3] Note that this is different from requiring that the $\beta_q^{(k)}$ themselves take maximal values!

Since this theorem depends on some fairly deep results in the formal theory, we will not present a proof but only discuss an algorithm to perform this completion. It is based on Theorem 3 above and consists essentially of two nested loops. The inner loop prolongs the system until its symbol becomes involutive. The outer loop checks then for integrability conditions and adds them. The difficult part of the proof is to show the termination of the inner loop. The termination of the outer one follows from a simple Noetherian argument.

Involution of a symbol can be checked easily using Definition 2, if we assume that the coordinate system is δ-regular what we will do in the sequel. It requires only linear algebra. Whether or not integrability conditions arise during a prolongation, can be deduced from a dimensional argument.

Denote the projection of the system $\mathcal{R}_{q+1}$ into the q-th order jet bundle $J_q\mathcal{E}$ by $\mathcal{R}_q^{(1)}$. Its dimension can be computed indirectly from the identity

$$\dim \mathcal{R}_q^{(1)} = \dim \mathcal{R}_{q+1} - \dim \mathcal{M}_{q+1} \tag{7}$$

which reflects the fact that integrability conditions are connected with rank defects in the symbol. None has occurred during the prolongation from $\mathcal{R}_q$ to $\mathcal{R}_{q+1}$, if and only if this dimension is equal to $\dim \mathcal{R}_q$.

There are essentially two possible reasons for integrability conditions. The classical one is that it is possible by some linear combination of equations of order $q+1$ in $\mathcal{R}_{q+1}$ to eliminate all derivatives of that order. This is a generalization of the usual cross-derivative. The other one is that $\mathcal{R}_q$ contains some equations of lower order. In order to construct $\mathcal{R}_{q+1}$ all equations in $\mathcal{R}_q$ must be prolonged. If now some equations are of lower order, it might happen that their prolongation leads to new independent equations of order less than or equal to q. They must be taken into account in the projection to $\mathcal{R}_q^{(1)}$.

Fig. 1 shows this algorithm in a more formal language. $\mathcal{R}_{q+r}^{(s)}$ denotes here the system obtained after $r+s$ prolongations and s projections. $\mathcal{M}_{q+r}^{(s)}$ is the corresponding symbol. In this form it is comparatively straightforward to implement it in a computer program. The determination of the dimensions of the various submanifolds $\mathcal{R}_{q+r}^{(s)}$ poses the main remaining problem, especially for non-linear systems. Ref. [27, 28] describe an implementation in the computer algebra system AXIOM.

For ordinary differential equations this algorithm becomes very simple. Since there is only one independent variable, we find always an involutive symbol and cross-derivatives are of course not possible. The only possibility for integrability conditions is the prolongation of lower order equations. For partial differential equations we recall that the other integrability conditions can always be found by considering the prolongations with respect to non-multiplicative variables.

To conclude this section we briefly recall some results of Ref. [29] concerning the arbitrariness of the general solution which will be needed later. (6) yields only the rank of the prolonged symbols, but their dimensions are more interesting. They can be expressed in a similar way, if we introduce the *Cartan*

[1]	r $\leftarrow$ 0; s$\leftarrow$ 0	
[2]	compute $\mathcal{R}_{q+1}$	$\{$*prolong*$\}$
[3]	compute $\mathcal{M}_q, \mathcal{M}_{q+1}$	$\{$*extract symbols*$\}$
[4]	`until` $\mathcal{R}^{(s)}_{q+r}$ involutive `repeat`	
[4.1]	`while` $\#multVar(\mathcal{M}^{(s)}_{q+r}) \neq \operatorname{rank} \mathcal{M}^{(s)}_{q+r+1}$ `repeat`	
[4.1.1]	r $\leftarrow$ r+1	$\{$*counter for prolongations*$\}$
[4.1.2]	compute $\mathcal{R}^{(s)}_{q+r+1}$	$\{$*prolong*$\}$
[4.1.3]	compute $\mathcal{M}^{(s)}_{q+r+1}$	$\{$*extract symbol*$\}$
[4.2]	`if` $\dim \mathcal{R}^{(s)}_{q+r+1} - \dim \mathcal{M}^{(s)}_{q+r+1} < \dim \mathcal{R}^{(s)}_{q+r}$ `then`	
[4.2.1]	s $\leftarrow$ s+1	$\{$*counter for projections*$\}$
[4.2.2]	compute $\mathcal{R}^{(s)}_{q+r}$	$\{$*add integrability conditions*$\}$
[4.2.3]	compute $\mathcal{R}^{(s)}_{q+r+1}$	$\{$*prolong*$\}$
[4.2.4]	compute $\mathcal{M}^{(s)}_{q+r}, \mathcal{M}^{(s)}_{q+r+1}$	$\{$*extract symbols*$\}$
[5]	`return` $\mathcal{R}^{(s)}_{q+r}$	

Fig. 1. Algorithm for the Cartan-Kuranishi Theorem

characters $\alpha^{(k)}_q$ of a differential equation

$$\alpha^{(k)}_q = m\binom{q+n-k-1}{q-1} - \beta^{(k)}_q, \qquad k = 1, \dots, n\,. \tag{8}$$

They form a descending sequence

$$\alpha^{(1)}_q \geq \alpha^{(2)}_q \geq \cdots \geq \alpha^{(n)}_q \geq 0\,. \tag{9}$$

Now we can write

$$\dim \mathcal{M}_{q+r} = \sum_{k=1}^{n} \alpha^{(k)}_{q+r} = \sum_{k=1}^{n} \binom{r+k-1}{r} \alpha^{(k)}_q\,. \tag{10}$$

This is the *Hilbert polynomial* of the differential equation $\mathcal{R}_q$ (it can be written explicitly as a polynomial in r). Analyzing the number of arbitrary Taylor coefficients in the power series expansion of the general solution and comparing with these dimensions yields the following result.

Theorem 5. *The general solution of a first-order system of differential equations $\mathcal{R}_q$ contains f_k functions depending on k arguments where the numbers f_k are determined by*

$$\begin{aligned} f_n &= \alpha^{(n)}_1 = m - \beta^{(n)}_1\,, \\ f_k &= \alpha^{(k)}_1 - \alpha^{(k+1)}_1 = \beta^{(k+1)}_1 - \beta^{(k)}_1\,. \end{aligned} \tag{11}$$

(9) ensures that the f_k are always non-negative. Note that Theorem 5 refers to algebraic representations of the general solution, i.e. no integrals or derivatives of the arbitrary functions do occur. One can derive more general results covering also higher-order equations and more general representations of the solution, but we will not need them here.

We define a *gauge symmetry* as a fiber-preserving transformation of the bundle $\mathcal{E}$ depending on some arbitrary functions of all independent variables which maps solutions into solutions. (This implies that f_n cannot vanish for a system with such a symmetry.) In gauge theories one identifies solutions related by a symmetry transformation. In order to obtain information about the arbitrariness of the physically relevant part of the solution space we must adjust the Cartan characters.

Let us assume that the gauge transformation can be written in the following form

$$\begin{aligned} \bar{x}^i &= \Omega^i(x^j)\,, \\ \bar{u}^\alpha &= \Lambda^\alpha\big(x^i, u^\beta, \lambda_a^{(0)}(x), \partial\lambda_a^{(1)}(x), \dots, \partial^p\lambda_a^{(p)}(x)\big) \end{aligned} \tag{12}$$

where γ_0 gauge functions $\lambda_a^{(0)}$ are entering algebraically, γ_1 gauge functions $\lambda_a^{(1)}$ are entering through their first derivatives etc. Ref. [28] shows how one can handle more general cases using a pseudogroup approach based on an implicit representation of the transformations by differential equations.

Under this assumption the gauge correction term $\Delta\alpha_q^{(k)}$ which must be subtracted from $\alpha_q^{(k)}$ to adjust for the effect of the symmetry can be computed recursively through

$$\Delta\alpha_q^{(k)} = \frac{(k-1)!}{(n-1)!}\sum_{l=0}^{p}\gamma_l s_{n-k-1}^{(n-1)}(q+l) - \sum_{i=k+1}^{n}\frac{(k-1)!}{(i-1)!}\Delta\alpha_q^{(i)} s_{i-k}^{(i-1)}(0)\,, \tag{13}$$

where the $s_k^{(n)}(q)$ denote some combinatorial factors, the modified Stirling numbers (earlier called symmetric q-products) introduced in Ref. [28, 29].

4 Constrained Dynamics à la Dirac

Let q^i be coordinates in some N-dimensional configuration space Q. We restrict our exhibition to autonomous systems, as explicit time dependencies can always be treated by considering the time as additional coordinate in an extended configuration space. The dynamics of a system is then determined by the condition that its action

$$S = \int L(q^i, \dot{q}^i)\,dt \tag{14}$$

is stationary along trajectories $q^i(t)$, where L is the Lagrangian of the system. It is well-known from the calculus of variations that this leads to the Euler-Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0\,, \qquad i = 1, \dots, N\,. \tag{15}$$

We pass from the Lagrangian formalism to the Hamiltonian one by a Legendre transformation. We introduce the canonically conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \,. \tag{16}$$

For regular systems the Legendre transformation provides a one-to-one mapping between the velocities $\dot{q}^i$ and the momenta p_i. In a constrained system this does no longer hold; instead one obtains by elimination some primary constraints

$$\phi_\ell(q^i, p_i) = 0 \,. \tag{17}$$

This implies that not every point of the phase space is accessible for the system (or can be used as initial data) but only a submanifold, i.e. some of the coordinates q^i do not correspond to true degrees of freedom.

The canonical Hamiltonian of the system given by

$$H_C = p_i \dot{q}^i - L(q^i, \dot{q}^i) \tag{18}$$

represents no longer the only possible choice. We can add arbitrary combinations of the constraints without changing its value on trajectories. This leads to the total Hamiltonian

$$H_T = H_C + u^\ell \phi_\ell \tag{19}$$

where the multipliers u^ℓ are a priori arbitrary functions of q^i, p_i.

The constraints must remain stable under the evolution of the system. Introducing the Poisson bracket

$$\Big\{ F(q^i, p_i), G(q^i, p_i) \Big\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i} \tag{20}$$

we can express the evolution of any observable $F(q^i, p_i)$ concisely

$$\dot{F} = \{F, H_T\} \,. \tag{21}$$

Thus we are lead to the requirement

$$\{\phi_\ell, H_T\} \approx 0 \,. \tag{22}$$

The sign $\approx$ signals that this is a so-called weak equality, it must hold only after taking all constraints into account. By a standard argument in differential geometry this implies that the Poisson bracket in (22) must be a linear combination of the constraints. There are three possibilities for (22): (i) it yields modulo the constraints an equation of the form $1 = 0$; (ii) it becomes $0 = 0$; (iii) we obtain a new equation $\psi(q^i, p_i) = 0$.

(i) means that our equations of motion are inconsistent. This implies that they do not possess any solution. Hence the Lagrangian is physically invalid. (ii) is of course the desired outcome. (iii) results in a secondary constraint. It is added to the other ones. We must of course then check whether all secondary constraints remain stable under the evolution of the system, i.e. we have to repeat the procedure until we either encounter case (i) or all constraints lead to case (ii). This is the so-called Dirac algorithm.

If secondary or higher constraints occur, we must distinguish whether or not they depend on the multipliers u^ℓ. If yes, we can solve for some of them which are then no longer arbitrary. This indicates the presence of second class constraints, as a first class constraint ψ Poisson commutes weakly with all other constraints ϕ_ℓ, i.e.

$$\{\psi, \phi_\ell\} \approx 0\,. \tag{23}$$

It is well-known that first class constraints generate gauge symmetries. Second class constraints correspond to unphysical degrees of freedom; a typical example is the pair $q^1 = 0$ and $p_1 = 0$. These unwanted degrees of freedom can be eliminated using the Dirac bracket. Let χ_ℓ denote all second class constraints and define the matrix C by

$$C_{ij} = \{\chi_i, \chi_j\}\ . \tag{24}$$

This matrix is always non-singular and we can define

$$\{f, g\}_* = \{f, g\} - \{f, \chi_i\}\,(C^{-1})^{ij}\,\{\chi_j, g\}\ . \tag{25}$$

In the canonical quantization of the system the Dirac brackets and not the Poisson brackets are transformed into commutation relations.

One of the fundamental goals in constrained dynamics is to count the number of degrees of freedom of the system. If there are N_F first and N_S second class constraints in the system, then the number F of dynamical degrees of freedom is given by

$$F = N - N_F - \frac{1}{2}N_S\,. \tag{26}$$

This simply reflects the fact that two second class constraints are necessary to eliminate a degree of freedom, as we need one for the coordinate and one for the momentum. A first class constraint leads to a symmetry and thus to an arbitrariness in a coordinate (or a momentum). A real elimination requires a gauge fixing condition, i.e. we add a new constraint which turns the first class constraint into a second class one.

5 Involution Analysis

Now we will analyze the equation of motions from the point of view of formal theory. Let the bundle $\mathcal{E}$ be given by $\mathcal{E} = Q \times T$, where Q is the N-dimensional configuration space with coordinates q^i and T the time axis, together with the natural projection $\mathcal{E} \to T$. (15) represents a second-order equation whose symbol is determined by the Hessian matrix

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\,. \tag{27}$$

If the symbol has rank N, the Euler-Lagrange Equations are normal and no constraints occur. Its general solution is parameterized by $2N$ arbitrary constants.

If, however, the symbol has lower rank, it is possible to eliminate the second-order derivatives in some of the equations. Now it is no longer obvious, whether or not (15) is involutive. Since we are dealing with an ordinary differential equation, the symbol is always involutive. But the prolongation of the obtained differential equations of lower order might lead to integrability conditions, if the resulting equations are independent of the remaining second-order equations in (15). Then we have to check whether some of these conditions are again of lower order; in that case we have to repeat the procedure.

After a finite number of iterations we will obtain either an inconsistency or an involutive system $\mathcal{R}_2^{(s)}$ of the following form

$$\mathcal{R}_2^{(s)} : \begin{cases} \ddot{q}^j = f^j(q^i, \dot{q}^i, \ddot{q}^n)\,, & j = 1, \dots, J, \quad n = J+1, \dots, N\,, \\ \dot{q}^k = g^k(q^i, \dot{q}^n)\,, & k = 1, \dots, K, \quad n = K+1, \dots, N\,, \\ q^m = h^m(q^n)\,, & m = 1, \dots, M, \quad n = M+1, \dots, N\,, \end{cases} \tag{28}$$

with $M \leq K \leq J$. Refs. [33, 34] contain detailed treatments of constrained systems in the Lagrangian formalism. A closer look reveals at once that it corresponds exactly to the completion algorithm presented in Section 3 applied to a system of second-order ordinary differential equations. Zeroth and first order equations are called there constraints of A and B type, respectively.

To relate our approach to the standard one by Dirac we pass again by a Legendre transformation to the Hamiltonian formulation. At the level of the differential equations this means that instead of the configuration space Q the phase space P is used to construct $\mathcal{E}$. In other words, we introduce N additional dependent variables p_i and transform (15) into the first-order equation

$$\mathcal{R}_1 : \begin{cases} \dot{p}_i = \dfrac{\partial L}{\partial q^i}\,, \\ p_i = \dfrac{\partial L}{\partial \dot{q}^i}\,, \end{cases} \qquad i = 1, \dots, N\,. \tag{29}$$

The first set of equations consists of course just of the Euler-Lagrange Equations with the second set of equations plugged in.

If the matrix M_{ij} has full rank, the second set of equations can be solved for the $\dot{q}^i$ and (29) is a normal equation. Otherwise we obtain some algebraic equations of the form $\phi_\ell(q^i, p_i) = 0$. They are of course the primary constraints of Dirac. Just as in the Lagrangian formulation, involution of (29) depends on the behavior of the prolonged equations

$$D_t\phi_\ell = \frac{\partial \phi_\ell}{\partial q^i}\dot{q}^i + \frac{\partial \phi_\ell}{\partial p_i}\dot{p}_i = 0\,. \tag{30}$$

We are only interested in these equations restricted to $\mathcal{R}_1$. This corresponds to the weak equalities used in the last section. Like in the Dirac algorithm there are three possibilities for the result of the restriction of each of the equations in (30): (i) it yields an inconsistency; (ii) it vanishes identically; (iii) we obtain a new independent equation.

If secondary constraints appear, we must repeat the procedure to check the consistency of the equations of motion (29). After a finite number of steps we will either have found an inconsistency or we will have constructed an involutive equation of the form

$$\mathcal{R}_1^{(s)} : \begin{cases} \dot p_i = \dfrac{\partial L}{\partial q^i}\,, & i = 1,\dots,N\,, \\ \dot q^j = f^j(q^i,\dot q^n,p_i)\,, & j = 1,\dots,M\,,\ n = M+1,\dots,N\,, \\ \phi_\ell(q^i,p_i) = 0\,, & \ell = 1,\dots,K\,. \end{cases} \tag{31}$$

In the Dirac approach one does not use the Hamiltonian equations (29), but one introduces some multipliers and takes those derived from the total Hamiltonian (19). To justify this we look at the differential of the canonical Hamiltonian (18)

$$dH_C = \dot q^i dp_i - \frac{\partial L}{\partial q^i}dq^i + \Big(p_i - \frac{\partial L}{\partial \dot q^i}\Big)d\dot q^i\,. \tag{32}$$

Thus on $\mathcal{R}_1$ we obtain

$$dH_C\Big|_{\mathcal{R}_1} = \dot q^i dp_i - \frac{\partial L}{\partial q^i}dq^i\,. \tag{33}$$

Two one-forms which coincide when restricted to the constraint surface, i.e. the submanifold of $\mathcal{E}$ defined by the constraints $\phi_\ell = 0$, can differ only by a linear combination of the form $u^\ell d\phi_\ell$ with arbitrary coefficients u^ℓ. Since

$$dH_C = \frac{\partial H_C}{\partial q^i}dq^i + \frac{\partial H_C}{\partial p_i}dp_i\,, \tag{34}$$

we obtain the following equations of motion (sometimes called Hamilton-Dirac equations) living in an extended phase space

$$\bar{\mathcal{R}}_1 : \begin{cases} \dot q^i = \dfrac{\partial H_C}{\partial p_i} + u^\ell\dfrac{\partial \phi_\ell}{\partial p_i}\,, \\ -\dot p_i = \dfrac{\partial H_C}{\partial q^i} + u^\ell\dfrac{\partial \phi_\ell}{\partial q^i}\,, \\ \phi_\ell(q^i,p_i) = 0\,. \end{cases} \tag{35}$$

Here the coefficients u^ℓ must be considered as additional functions of t or in the language of differential equations as additional dependent variables. (29) is obtained, if we use the first set of equations in (35) to express u^ℓ through $q^i, \dot q^i$. Thus both systems are equivalent.

The Dirac algorithm is equivalent to the completion of system (35). It requires the analysis of the prolongations of the constraints restricted to $\bar{\mathcal{R}}_1$. They can be concisely written using Poisson brackets

$$\begin{aligned} D_t\phi_\ell\Big|_{\bar{\mathcal{R}}_1} &= \frac{\partial\phi_\ell}{\partial q^i}\frac{\partial H_C}{\partial p_i} - \frac{\partial\phi_\ell}{\partial p_i}\frac{\partial H_C}{\partial q^i} + u^k\left(\frac{\partial\phi_\ell}{\partial q^i}\frac{\partial\phi_k}{\partial p_i} - \frac{\partial\phi_\ell}{\partial p_i}\frac{\partial\phi_k}{\partial q^i}\right) \\ &= \Big\{\phi_\ell, H_C\Big\} + u^k\Big\{\phi_\ell,\phi_k\Big\}\,. \end{aligned} \tag{36}$$

Actually, to obtain a full equivalence we should write the multiplier as derivatives $u^\ell = \dot{v}^\ell$. This is only important in the case that second class constraints are present. Then we obtain equations determining some of the u^ℓ. In principle we must prolong then these equations, too. This unnecessary step which yields no new information can be omitted by using derivatives. This appears also from a physical point of view somewhat more natural, as the multipliers correspond to velocities.

6 Counting Degrees of Freedom

The classical expression (26) for the number of degrees of freedom depends on the distinction into first and second class constraints. This requires, however, the introduction of a Poisson structure. We will show now that it is possible to obtain an intrinsic expression for this number without performing such a distinction.

We will start with the Hamiltonian equations of motion (31). In order to obtain the number of degrees of freedom we must count the number of constants necessary to characterize a physical state. Since $\dim J_1\mathcal{E} = 4N$, we obtain $\dim\mathcal{R}_1^{(s)} = 3N - M - K$. Thus in a power series expansion of the general solution this number of zeroth and first order coefficients can be chosen arbitrarily.

If we identify the arbitrary functions with $\dot{q}^i$ for $i > M$, as these are not restricted by (31), we must subtract $N - M$ constants coming from the arbitrary functions. Thus an initial state at $t = t_0$ is specified by $2N - K$ constants. Depending on the choice of the arbitrary functions we will, however, obtain different values for $q^i(t_1)$ and $p_i(t_1)$ at some later instant t_1. As these correspond nevertheless, by definition, to the same physical state, we must subtract further $N - M$ constants for the gauge symmetry. Thus a physical state is specified by $N + M - K$ constants and the number of degrees of freedom F is half of this number

$$F = \frac{1}{2}(N + M - K)\,. \tag{37}$$

We find for $\mathcal{R}_1^{(s)}$ that $\beta_1^{(1)} = N + M$ and hence $\alpha_1^{(1)} = N - M$. Expressing M by N and $\alpha_1^{(1)}$ and similarly K by N, $\alpha_1^{(1)}$ and $\dim\mathcal{R}_1^{(s)}$ yields an intrinsic expression for F independent of any specific representation of the manifold $\mathcal{R}_1^{(s)}$

$$F = \frac{1}{2}\dim\mathcal{R}_1^{(s)} - \alpha_1^{(1)}\,. \tag{38}$$

If we use instead of the Hamiltonian equations of motion (29) the Euler-Lagrange Equations (15), we obtain the analogue result

$$F = \frac{1}{2}\dim\mathcal{R}_2^{(s)} - \alpha_2^{(1)} . \tag{39}$$

Both expressions yield always the same value, as we will obtain exactly the same dimension and Cartan character for the final involutive equation no matter whether we work in first- or second-order because of the different dimensions of the base spaces [23, 28].

As a by-product this implies that J in (28) equals M in (31) and the sum of K and M in (28) equals K in (31). Thus we have the same number of constraints in both approaches, if we omit the introduction of multipliers. This might not be very surprising, we will, however, see later that this does no longer hold in field theories.

We hope to study in a future paper the distinction into first and second class constraints in more detail. Then we will also be able to discuss the relation between these results and the classical formula (26). For the moment we just note that according to Theorem 5 the general solution contains $\alpha_1^{(1)}$ arbitrary functions. In the classical terminology this arbitrariness stems from the gauge transformations generated by the primary first class constraints [14, 15]. Thus their number is $\alpha_1^{(1)}$.

7 Example

Consider the classical problem of a particle whose movement is restricted to the surface of a sphere in a D-dimensional space but otherwise free, often also called the rigid rotator [9]. Without loss of generality we can take the radius of the sphere as one and start with the Lagrangian

$$L(q^i, \dot{q}^i, \lambda, \dot{\lambda}) = \frac{m}{2}\dot{q}^2 + \lambda(q^2 - 1) . \tag{40}$$

($q^2 = q^i q_i$, etc.) λ is here obviously a multiplier. The canonically conjugate momenta p_i, π are given by

$$\begin{aligned} p_i &= m\dot{q}_i , \quad i = 1, \dots, D , \\ \pi &= 0 . \end{aligned} \tag{41}$$

If we introduce a further multiplier μ, we can write the total Hamiltonian as

$$H_T = \frac{1}{2m}p^2 - \lambda(q^2 - 1) + \mu\pi \tag{42}$$

Obviously there is one primary constraint, namely $\pi = 0$. The next three steps of the Dirac algorithm lead to the constraints $q^2 = 1$, $pq = 0$, and finally $p^2 = -2m\lambda$. It is easy to see that all these constraints are second class. The system contains thus $D - 1$ degrees of freedom which can be formally calculated by subtracting from the dimension of the configuration space — D coordinates q_i plus one coordinate λ — half the number of second class constraints, i.e. 2.

Now we will obtain the number of degrees of freedom using a formal analysis of the Euler-Lagrange Equations

$$\mathcal{R}_2 \,:\, \begin{cases} m\ddot{q}_i - 2\lambda q_i = 0\,, \\ q^2 - 1 = 0\,. \end{cases} \tag{43}$$

The completion to an involutive equation requires four projections leading to the integrability conditions $q\dot{q} = 0$, $q\ddot{q} + \dot{q}^2 = 0$, $\dot{\lambda} = 0$ and finally $\ddot{\lambda} = 0$. After some trivial manipulations we have thus the involutive equation

$$\mathcal{R}_2^{(4)} \,:\, \begin{cases} m\ddot{q}_i - 2\lambda q_i = 0\,, & \ddot{\lambda} = 0\,, \\ q\dot{q} = 0\,, & \dot{\lambda} = 0\,, \\ q^2 - 1 = 0\,, & m\dot{q}^2 + 2\lambda = 0\,. \end{cases} \tag{44}$$

It is easy to see that this represents a finite type equation and thus there are no first class constraints. Since $\dim \mathcal{R}_2^{(4)} = 2D - 2$, (38) yields $F = D - 1$ degrees of freedom, in perfect agreement with the Dirac analysis.

Alternatively, we can analyze the Hamiltonian equations of motion

$$\mathcal{R}_1 \,:\, \begin{cases} \dot{p}_i - 2\lambda q_i = 0\,, & \dot{\pi} - q^2 + 1 = 0\,, \\ m\dot{q}_i - p_i = 0\,, & \pi = 0\,. \end{cases} \tag{45}$$

Again the system becomes involutive after four projections with integrability conditions $\dot{\pi} = 0$, $qp = 0$, $p^2 + 2m\lambda = 0$, and finally $\dot{\lambda} = 0$. This yields

$$\mathcal{R}_1^{(4)} \,:\, \begin{cases} \dot{p}_i - 2\lambda q_i = 0\,, & \dot{\pi} = 0\,, \\ m\dot{q}_i - p_i = 0\,, & \dot{\lambda} = 0\,, \\ \pi = 0\,, & p^2 + 2m\lambda = 0\,, \\ qp = 0\,, & q^2 - 1 = 0\,. \end{cases} \tag{46}$$

The analysis of this equation leads of course to exactly the same number of degrees of freedom, as $\dim \mathcal{R}_1^{(4)} = \dim \mathcal{R}_2^{(4)}$ and both are finite type equations.

8 Field Theories

In the section on point mechanics we studied three different ways to write the equations of motion: the Lagrangian equations, the Hamiltonian equations obtained from the latter one by direct application of the Legendre transformation, and finally the Hamilton-Dirac equations where one includes multipliers for the primary constraints. In field theories there is even more choice, as there exist at least two different ways to perform the Legendre transformation.

The standard way entails the explicit choice of a time variables and leads to a non-covariant formalism. From the point of view of differential equations this approach has a further disadvantage: it is not a truly first-order formalism, as the Hamiltonian field equations will generally still contain second-order spatial derivatives. Thus for the application of involution theory it is probably more appropriate to use the so-called De Donder-Weyl approach [16, 25] which leads to a covariant first-order formalism.

We will therefore restrict ourselves in the sequel to the analysis of field equations in Lagrangian formulation and leave the discussion of the Hamiltonian approach for the future. This suffices for the purpose of this article.

Many articles on the theory of systems with constraints have the following structure: The theoretical results are derived in the finite-dimensional case, i.e. in point mechanics; the examples and applications stem, however, from field theories. The connection is made with a remark like "The generalization of these results to field theories is straightforward". But this point of view is a bit optimistic, as a more careful discussion (see e.g. [34]) reveals many subtle problems.

Although on the surface the main difference lies in the fact that Poisson brackets are now computed via functional derivatives instead of partial ones, many elementary concepts in the finite-dimensional theory become rather tricky in an infinite-dimensional setting. For instance linear combinations must now be substituted by integrations. But to require that an integral vanishes is a much less stringent condition than the vanishing of an algebraic expression and depends decisively on the considered function space.

Similarly, inverses as they are used in the construction of Dirac brackets are no longer uniquely defined. Often already the distinction into first and second class constraints can be rather problematic and statements like the number of second class constraints is always even do not make sense any more. One must introduce the new concept of proper and improper constraints [1]. Further problems stemming from the choice of coordinates will be discussed in Section 10.

To really solve these problem one must usually resort to fairly complicated methods from functional analysis. We will concentrate in this section, however, on another point: The naive generalization of the Dirac analysis does *not* correspond to the completion to involution of the field equations. Thus in general it does not suffice to prove their consistency.

Since field equations are partial differential equations, involution becomes a more complicated concept. The prolongation of lower order equations represents no longer the only way to generate integrability conditions. Constraints are mostly equations of lower class. In a typical field theory the base space X of independent variables is a D-dimensional space-time. We can identify the variable x^D with time. Thus in the usual terminology, the equations of class D are the evolutionary ones; the remaining ones are constraints.

The naive generalization of the Dirac algorithm prolongs all constraints only with respect to time, as it relies solely on Poisson brackets with the Hamiltonian. If all constraints are of class $D-1$, this corresponds to our approach, because it suffices to analyze the prolongations with respect to the non-multiplicative variables and we find in this case only one, namely x^D.

But now the question arises as to what happens if constraints of lower class appear. Then prolongations with respect to the other multiplicative variables, i.e. with respect to some spatial coordinates, may lead to additional integrability conditions not considered by this naive approach.

In order to exhibit this effect in "pure form" we begin with a highly unphysical example without any kinetic term in the Lagrangian density. But we will later show that this is not the important point. Consider the class of systems described by the Lagrangian density

$$\mathcal{L}[\phi,\lambda,\mu] = \mu[\partial_x\phi - f(\phi)] + \lambda[\partial_y\phi - g(\phi)] \tag{47}$$

on a three-dimensional flat space-time with coordinates x, y, t. The fields λ, μ represent again multipliers, whereas f, g denote fixed but so far arbitrary functions. Variation with respect to ϕ yields the equation

$$\partial_x\mu + \partial_y\lambda + f'(\phi)\mu + g'(\phi)\lambda = 0\,. \tag{48}$$

More interesting equations are obtained from the multipliers. They generate an over-determined system for ϕ

$$\begin{aligned} \partial_x\phi - f(\phi) &= 0\,,\\ \partial_y\phi - g(\phi) &= 0\,. \end{aligned} \tag{49}$$

Obviously this system is consistent, if and only if f, g satisfy the equation

$$f'g = fg'\,. \tag{50}$$

This requires that f is a multiple of g. It is easy to see that only under this condition the Euler-Lagrange Equations are involutive. We must conclude that most of the Lagrangian densities (47) are physically invalid.

Now we look at the outcome of the naive Dirac algorithm applied to this field theory. Obviously all three canonically conjugate momenta $\pi_\phi, \pi_\mu, \pi_\lambda$ vanish and represent thus primary constraints. The total Hamiltonian density is given by

$$\mathcal{H}_T = -\mu[\partial_x\phi - f(\phi)] - \lambda[\partial_y\phi - g(\phi)] + u\pi_\phi + v\pi_\mu + w\pi_\lambda\,. \tag{51}$$

This yields the following secondary constraints

$$\begin{aligned} \{\pi_\phi, \mathcal{H}_T\} &= -[\partial_x\mu + \partial_y\lambda + f'(\phi)\mu + g'(\phi)\lambda]\,,\\ \{\pi_\mu, \mathcal{H}_T\} &= \partial_x\phi - f(\phi)\,,\\ \{\pi_\lambda, \mathcal{H}_T\} &= \partial_y\phi - g(\phi)\,. \end{aligned} \tag{52}$$

Thus we obtain exactly the Euler-Lagrange Equations above. But note the crucial difference in the further analysis. Following the naive Dirac analysis we look only whether these equations are *algebraically* related, i.e. whether one vanishes, if we take the others into account. But this does not happen here.

Since all constraints are second class, we continue to compute the tertiary constraints in order to fix the multipliers u, v, w introduced in the total Hamiltonian density. They are

$$\begin{aligned} &\partial_x v + \partial_y w + f''(\phi)\mu u + f'(\phi)v + g''(\phi)\lambda u + g'(\phi)w = 0\,, \\ &\partial_x u - f'(\phi)u = 0\,, \\ &\partial_y u - g'(\phi)u = 0\,. \end{aligned} \tag{53}$$

We find the interesting phenomenon that although there are six second class constraints for three degrees of freedom ϕ, μ, λ, they do not fix all multipliers. v, w appear only in the first equation, hence one of them can be chosen arbitrarily. u is the solution of an over-determined system which is consistent, if f and g satisfy (50).

This example shows that the real failure of this approach lies in the purely algebraic treatment of the constraints. In point mechanics we could always substitute one constraint in another one, if both contained the same coordinate q^i or a (time) derivative of it. This is no longer possible in field theories, because there might be derivatives with respect to other coordinates present. To check the consistency of such constraints requires the analysis of integrability conditions.

This analysis is surely trivial in a primitive example as above. But in more complicated cases it is rather difficult to decide when one has found all integrability conditions. Assume for instance that a field theory in 3+1 dimensions leads (among others) to the following constraints for some field ϕ

$$\begin{aligned} \phi_{zz} + y\phi_{xx} &= 0\,, \\ \phi_{yy} &= 0\,. \end{aligned} \tag{54}$$

In this famous example due to Janet one needs five prolongations and two projections to show that there are exactly two integrability conditions, namely $\phi_{xxy} = \phi_{xxxx} = 0$. Such problems lead at the end of the last century to the first steps towards the development of the formal theory!

One might argue that (47) is a rather peculiar Lagrangian density. But the addition of a kinetic term for ϕ does not really change the outcome, although the computations are slightly more complicated due to the appearance of further integrability conditions. Take

$$\bar{\mathcal{L}} = \frac{1}{2}(\partial_t\phi)^2 + \mathcal{L}\,. \tag{55}$$

Its Euler-Lagrange Equations are

$$\begin{aligned} &\partial_{tt}\phi + \partial_x\mu + \partial_y\lambda + f'(\phi)\mu + g'(\phi)\lambda = 0\,, \\ &\partial_x\phi - f(\phi) = 0\,, \\ &\partial_y\phi - g(\phi) = 0\,. \end{aligned} \tag{56}$$

Only the first equation has changed. But since the only second-order derivative involves ϕ, our completion procedure generates besides (50) two further integrability conditions involving second-order derivatives of μ, λ. We omit them here, because they are rather complicated.

We also do not show the Dirac analysis of this system. It suffices to note that the two new conditions (in Hamiltonian form) are also found by the Dirac algorithm. But this was to be expected, because their construction requires prolongations with respect to time. Only (50) is again overlooked, for no spatial cross-derivatives are performed.

The natural question is whether systems of this form are unphysical for some reason or whether this effect occurs often and might lead to a wrong results. Such purely spatial cross-derivatives (or to be more precise: linear combinations of spatial prolongations) are only necessary, if the field equations form an overdetermined system. This might appear at first sight quite unusual but one can find such systems in the literature (see e.g. [11]). A typical example is a gauge fixing condition built into the Lagrangian density.

9 Higher-Order Lagrangians

Theories described by Lagrangian containing higher order derivatives [3, 13, 20, 24] can be treated in exactly the same way. Using again the notation of Section 2 such a Lagrangian is a function $L(x^i, u^\alpha, p^\alpha_\mu)$. The x^i are coordinates on the underlying space-time in the case of a field theory or just the time in point mechanics. Similarly, the u^α denote either the fields in the theory or the generalized coordinates and the p^α_μ represent the derivatives.

From the calculus of variations it is well-known that the Euler-Lagrange Equations can now be written using the Euler operators [21]

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} + \sum_\mu (-1)^{|\mu|} D_\mu \left(\frac{\partial L}{\partial p^\alpha_\mu}\right) = 0 \,. \tag{57}$$

If $\mu = [\mu_1, \ldots, \mu_n]$, then $D_\mu = D_1^{\mu_1} \cdots D_n^{\mu_n}$ where D_i denotes the total derivative with respect to x^i. Obviously we recover (15), if there is only one x^i, namely the time t, and L depends only on derivatives with $|\mu| = 1$. The sum in (57) is always finite, as L contains only derivatives up to a given order.

As in the standard case there exist at least three possibilities for the starting point of the involution analysis. The simplest choice is of course to use directly the Euler-Lagrange Equations (57). Alternatively one can pass to a Hamiltonian formulation. This requires now the introduction of several momenta conjugate to each field. Then one can either transform the Euler-Lagrange Equations directly or one can derive in the same manner as before the Hamilton-Dirac Equations by introducing multipliers.

There is no need to repeat the arguments of Section 5, as they still apply in the same way. Instead we consider as an example Podolsky's generalized electrodynamics [22] in the Lagrangian formalism. It demonstrates the typical way an involution proof proceeds for field theories. Refs. [28, 30] contain further examples like Yang-Mills or Einstein Equations. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - a^2\partial_\rho F^{\sigma\rho}\partial_\tau F^\tau_\sigma \,. \tag{58}$$

The space-time indices run from 1 to D; we identify x^D with the time. a is a constant. If it vanishes, we recover the standard Maxwell theory.

In terms of the vector potential A_μ the Euler-Lagrange Equations (57) give a fourth-order system

$$\mathcal{R}_4 : \quad (1-2a^2\Box)\Box A_\mu - \partial_\mu\left[(1-2a^2\Box)\partial^\nu A_\nu\right] = 0\,, \quad \mu = 1,\dots,D \tag{59}$$

with the D'Alembertian $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

According to Definition 2 we must first check whether the symbol $\mathcal{M}_4$ is involutive. It is given by the equations

$$\mathcal{M}_4 : \quad \eta^{\mu\nu}\eta^{\rho\sigma}v^\alpha_{\mu\nu\rho\sigma} - \eta^{o\tau}\eta^{\alpha\beta}v^\delta_{o\tau\beta\delta} = 0\,, \quad \alpha = 1,\dots,D \tag{60}$$

where $v^\alpha_{\mu\nu\rho\sigma}$ is a placeholder for the derivative $\partial_{\mu\nu\rho\sigma}A^\alpha$. It is easy to see that for $\alpha \neq D$ we can choose in each equation the variable v^α_{DDDD} as pivot, i.e. all these equations are of class D. For $\alpha = D$, however, the corresponding variable cancels. We can obtain at most a pivot of class $D-1$, e.g. $v^D_{D-1,D-1,D-1,D-1}$. Thus

$$\beta_4^{(D)} = D-1\,, \quad \beta_4^{(D-1)} = 1\,, \quad \beta_4^{(k)} = 0\,. \tag{61}$$

One can prove that this cancellation for $\alpha = D$ does not simply stem from a singular coordinate system either by using the tableau of the system (cf. [28]) or by arguing that $\beta_4^{(D)} = D$ is not possible, as the system has a gauge symmetry [10]. This argument relies on the results of Ref. [29] on the arbitrariness of the general solution.

In order to apply Definition 2 we must compute next the rank of the prolonged symbol $\mathcal{M}_5$. It is defined by

$$\mathcal{M}_5 : \quad \eta^{\mu\nu}\eta^{\rho\sigma}v^\alpha_{\mu\nu\rho\sigma\gamma} - \eta^{o\tau}\eta^{\alpha\beta}v^\delta_{o\tau\beta\gamma\delta} = 0\,, \quad \alpha,\gamma = 1,\dots,D\,. \tag{62}$$

It is easy to see that these equations are not all independent, because if we set $\alpha = \gamma$ and sum the result vanishes.

It follows from the discussion before Definition 2 and the obtained values for the $\beta_4^{(k)}$ that $\operatorname{rank}\mathcal{M}_5 \geq D^2-1$. But since we have already found an identity, the same value represents an upper bound for the rank. Thus the rank must be exactly D^2-1 and by (4) $\mathcal{M}_4$ is involutive.

In order to see whether integrability conditions occur we must check whether this identity holds only at the level of the prolonged symbol or also if we use the full prolonged equations. We know already that the fifth order derivatives cancel, but it could happen that some lower order equation remains. It is, however, easy to see that this is not the case. This is analogue to the Noether identity in the Maxwell theory. Hence $\mathcal{R}_4$ is involutive. This implies that in the Lagrangian formalism only one constraint appears, namely the equation for $\alpha = D$ which is of lower class.

For later use we note the Cartan characters of the theory

$$\alpha_4^{(4)} = 1\,, \quad \alpha_4^{(3)} = 15\,, \quad \alpha_4^{(2)} = 40\,, \quad \alpha_4^{(1)} = 80\,. \tag{63}$$

Adjusting for the symmetry $A_\mu \to A_\mu + \partial_\mu \lambda$ with (13) yields the following gauge corrected values ($\gamma_1 = 1$)

$$\bar{\alpha}_4^{(4)} = 0\,, \quad \bar{\alpha}_4^{(3)} = 10\,, \quad \bar{\alpha}_4^{(2)} = 25\,, \quad \bar{\alpha}_4^{(1)} = 45\,. \tag{64}$$

The Hamiltonian treatment is similar but more involved, as now secondary constraints appear. There is no need to detail it here. Since the only constraint in the Lagrangian formalism is of class $D-1$, the Dirac analysis is sufficient and equivalent to the involution analysis. Thus we just recover the calculations presented in Ref. [10].

The constraint analysis of Podolsky's generalized theory is very similar to the standard Maxwell theory. In both cases we find that the Euler-Lagrange Equations are already involutive due to the Noether identity, whereas in the Hamiltonian formalism we must perform a few steps until we reach an involutive system. The same effect can be observed in other field theories. This seems to imply that at least at classical level the Lagrangian formalism is more efficient, as it yields faster an involutive system.

10 "Field Theoretical" Degrees of Freedom

The classical procedure to count degrees of freedom in field theories is simply to stick to the rule (26) used in the finite-dimensional case. N denotes now the number of fields. The argument is essentially the same: Each constraint "fixes" one field in the phase space and in the case of a first class constraint the symmetry eliminates a further degree of freedom.

The problem with this approach is that in field theories constraints are usually differential equations. Hence they cannot really fix a field; there remains some freedom. Speaking about arbitrary functions in the general solution the idea behind the rule above seems to be that a degree of freedom corresponds to the possibility to prescribe as initial data a function of $D-1$ arguments, i.e. of all spatial variables. This is for instance the case in a regular theory, as its field equations form a normal system satisfying the conditions of the Cauchy-Kowalevsky Theorem [2].

A generalization of this idea was already introduced by Einstein [7] with his definition of the strength of a system of differential equations. Up to a numerical factor depending on the dimension of space-time it can be identified with the number of arbitrary functions of $D-1$ arguments [29] provided the system is absolutely compatible. Latter condition entails that there are no arbitrary functions of D arguments.

Of course such considerations make sense only after taking gauge symmetries into account, because such a symmetry leads to arbitrary functions of all independent variables. Einstein's definition of the strength contains such correction terms. Based on the formal theory of partial differential equations we propose as an intrinsic definition for the number of "field theoretical" degrees of freedom the gauge corrected Cartan character $\bar{\alpha}_q^{(D-1)}$. This definition covers also theories derived from Lagrangians containing higher-order derivatives.

According to Theorem 5 $\bar{\alpha}_q^{(D-1)}$ corresponds to the number of arbitrary functions of $D-1$ arguments in the general solution modulo the gauge symmetry. Of course we assume here that $\bar{\alpha}_q^{(D)} = 0$ as in Einstein's approach. Otherwise either the field equations are under-determined or the gauge group was not correctly identified. For a regular theory no gauge correction is needed. In this case $\alpha_q^{(D)} = 0$ and $\alpha_q^{(D-1)} = m$ where m is the number of fields. Thus we recover the usual result.

For the Maxwell theory one obtains the following gauge corrected Cartan characters [28]

$$\bar{\alpha}_2^{(4)} = 0\,, \quad \bar{\alpha}_2^{(3)} = 4\,, \quad \bar{\alpha}_2^{(2)} = 6\,, \quad \bar{\alpha}_2^{(1)} = 0\,. \tag{65}$$

(The same values can be obtained without gauge correction by directly analyzing the field equations in field strength formulation.) Thus we obtain in perfect agreement with the usual result 4 degrees of freedom. In contrast Podolsky's generalized theory possesses 10 degrees of freedom as evident from (64).

It is important here to note that the Cartan characters are intrinsically defined and thus independent of any specific coordinate system. This is of course a property one should expect of a reasonable definition for the number of degrees of freedom. As Steinhardt [32] showed with some explicit examples (see also the discussion in [34]), the classical approach encounters problems, if "wrong" coordinates are used.

He considered among others the simple example of a free massive scalar field in 1+1 dimensions. In standard coordinates this system described by the Lagrangian density $2\mathcal{L} = -\partial^\mu\phi\partial_\mu\phi - m^2\phi^2$ is obviously regular and contains one degree of freedom. In light-cone coordinates $x^\pm = (x \pm t)/2$ the Lagrangian density becomes

$$\mathcal{L} = \partial_+\phi\partial_-\phi - m^2\phi^2/2\,. \tag{66}$$

If we choose x^+ as new evolution parameter, the canonically conjugate momentum is $\pi = \partial_-\phi$ and independent of the velocity $\partial_+\phi$. Hence the system is constrained.

It is quite subtle to decide whether this constraint is first or second class, but here we are not concerned with these difficulties. The important fact is that there appears a constraint and hence according to (26) the system has less than one degree of freedom! Obviously this cannot be correct. We are not aware of any proposal in the literature for a modified formula to count degrees of freedom that takes this effect into account.

In the involution analysis a similar phenomenon occurs. As already Steinhardt pointed out, the appearance of the constraint is intimately connected with the fact that the light-cone coordinates are the characteristics of the field equations. Actually this represents a special case of a more general problem, namely the δ-regularity of a coordinate system [23] which was already mentioned in Section 2.

If a coordinate system is not δ-regular, the procedure to compute the $\beta_q^{(k)}$ described in Section 2 yields too small values. This corresponds to the too many constraints found by Steinhardt. For instance in our case the field equations are

$$2\partial_+\partial_-\phi + m^2\phi = 0\,. \qquad (67)$$

Obviously there is no derivative of class 2 in this equation. But it can be generated with a simple coordinate transformation, namely going back to the original ones. Thus the correct value for $\beta_2^{(2)}$ is one and not zero. This yields $\alpha_2^{(2)} = 0$ and $\alpha_2^{(1)} = 1$. Since no gauge correction is necessary here, we obtain the expected result: one degree of freedom.

There exists a simple method to determine the correct values of the $\beta_q^{(k)}$ in *any* coordinate system without performing a coordinate change. It makes use of the generalized tableaux of a differential equation. Their ranks provide an intrinsic definition for the $\beta_q^{(k)}$; i.e. their determination is a simple problem in linear algebra. For lack of space we cannot detail this approach here but refer to the literature [31, 28]. The important point is that our definition for the number of degrees of freedom can be applied to any system in any coordinate system and leads always to the same number.

11 Conclusion

Obviously it is one of the most elementary requirements on a system of differential equations to be consistent, i.e. to possess at least a formal power series solution. The formal theory of differential equations provides us with a powerful tool to check this property: the involution analysis. We have shown how it can be used in a physical context, namely in constrained dynamics. Here the differential equations arise typically as the equation of motions derived from some Lagrangian L.

One should keep in mind that the motivation for the Dirac analysis is exactly the same. Its first task is less to exhibit all constraints but to prove the consistency of the equations of motion. Hence it is not very surprising that we find that for finite-dimensional systems the Dirac analysis coincides with the involution analysis. Both approaches yield (in some sense merely as a by-product) all constraints or integrability conditions, respectively, of the system.

Although it seems to be a commonly accepted claim that the Dirac analysis can be extended without modifications to field theories, we have given examples where in our opinion this classical approach is not sufficient. Their construction was based on the simple observation that Dirac takes only the temporal evolution of the system into account. He does not consider spatial prolongations. Hence his approach is incomplete and not able to prove the consistency of the field equations.

One cannot really speak of a failure of the Dirac approach, but one must note that it must be augmented by some kind of analysis of the spatial integrability conditions. One could for instance use an approach like the involution analysis for this purpose. But we think that it is conceptually easier to use one method instead of a combination of several ones.

We believe that the involution approach is more flexible than the Dirac method. It can be applied to the Lagrangian as well as to the Hamiltonian formalism. In a future article we will show that the so-called symplectic method for first-order Lagrangian [8] also fits into this scheme. The same holds for higher-order Lagrangians. As soon as the equations of motion are obtained, either by a variational principle or some other way, the involution analysis can start. This is especially important for systems with anholonomic constraints which cannot be treated with the usual variational methods [25].

An important advantage of the involution analysis is that it represents a geometric framework, i.e. all constructions are intrinsic and coordinate independent. This is not true for the Dirac approach. It is well-known that the number of constraints can be different in different coordinate systems. A typical example are light-cone coordinates. In contrast, our definition of the number of degrees of freedom for a field theory via the Cartan characters is completely intrinsic. To our knowledge no such definition has so far been proposed in the literature.

One might wonder where all the subtleties of the Dirac analysis like regularity conditions on the constraints, ineffective or reducible constraints have disappeared [15]. They are of course still present. Most of them are hidden behind the calculation of the dimension of the various submanifolds $\mathcal{R}_{q+r}^{(s)}$ used in the completion algorithm depicted in Fig. 1. But this is a classical problem in geometry and for special types of constraints there may exist alternative approaches. For instance polynomial non-linearities represent probably the most important case in applications. For them we can avoid a discussion of most of the mentioned problems by using Gröbner bases techniques [17].

Another effect which might lead to problems is that the rank of the symbol (or more generally the numbers $\beta_q^{(k)}$) does not need to be constant. It might change, if certain additional differential equations hold. This generalizes the classification of the eigenvalues of the Hessian of the Lagrangian as given by Lusanna [19].

Finally, one should note that by making contact with the formal theory of differential equations one obtains suddenly a well understood object, namely an involutive system. Many properties of such systems are known and many techniques have been developed for their further analysis. All these results are now available for constrained systems.

Acknowledgments

We thank M. Henneaux for helpful comments. WMS is supported by a grant of the School of Physics and Materials, Lancaster University and RWT is grateful for support by a grant in the EC Human Capital and Mobility program.

References

1. R. Benguria, P. Cordero, and C. Teitelboim. Aspects of the Hamiltonian dynamics of interacting gravitational gauge and Higgs fields with applications to spherical symmetry. *Nucl. Phys. B*, 122:61–93, 1977.
2. R. Courant and D. Hilbert. *Methods of Mathematical Physics*, volume 2. Interscience Publishers, New York, 1962.
3. T. Dereli, D. Hartley, M. Önder, and R.W. Tucker. Relativistic elastica. *Phys. Lett. B*, 252:601–604, 1990.
4. P.A.M. Dirac. Generalized Hamiltonian dynamics. *Can. J. Math.*, 2:129–148, 1950.
5. P.A.M. Dirac. Generalized Hamiltonian dynamics. *Proc. Roy. Soc. A*, 246:326–332, 1958.
6. P.A.M. Dirac. *Lectures on Quantum Mechanics.* Belfer Graduate School Monograph Series 3. Yeshiva University, New York, 1964.
7. A. Einstein. *The Meaning of Relativity.* Princeton University Press, Princeton, 5th edition, 1955.
8. L. Faddeev and R. Jackiw. Hamiltonian reduction of unconstrained and constrained systems. *Phys. Rev. Lett.*, 60:1692–1694, 1988.
9. N.K. Falck and A.C. Hirshfeld. Dirac-bracket quantization of a constrained nonlinear system: The rigid rotator. *Eur. J. Phys.*, 4:5–9, 1983.
10. C.A. Galvão and B.M. Pimentel. The canonical structure of Podolsky generalized electrodynamics. *Can. J. Phys.*, 66:460–466, 1988.
11. M.J. Gotay. On the validity of Dirac's conjecture regarding first-class secondary constraints. *J. Phys. A*, 16:L141–L145, 1983.
12. A. Hanson, T. Regge, and C. Teitelboim. *Constrained Hamiltonian Systems.* Academia Nazionale dei Lincei, Rome, 1976.
13. D. Hartley and R.W. Tucker. On the motion of a charged relativistic particle with spin. *Phys. Lett. A*, 159:193–197, 1991.
14. D. Hartley, R.W. Tucker, and P.A. Tuckey. Constrained dynamics and exterior differential systems. *J. Phys. A*, 24:5253–5265, 1991.
15. M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems.* Princeton University Press, 1992.
16. I.V. Kanatchikov. Basic structures of the covariant canonical formalism for fields based on the De Donder-Weyl theory. Technical Report PITHA 94/47, Universität Aachen, 1994.
17. H. Kredel and V. Weispfenning. Computing dimension and independent sets for polynomial ideals. *J. Symb. Comp.*, 6:231–247, 1988.
18. M. Kuranishi. On E. Cartan's prolongation theorem of exterior differential systems. *Amer. J. Math.*, 79:1–47, 1957.
19. L. Lusanna. The second Noether theorem as the basis of the theory of singular Lagrangians and Hamiltonian constraints. *Riv. Nuovo Cim.*, 14(3):1–75, 1991.
20. V.V. Nesterenko. Singular Lagrangians with higher derivatives. *J. Phys. A*, 22:1673–1687, 1989.
21. P.J. Olver. *Applications of Lie Groups to Differential Equations.* Graduate Texts in Mathematics 107. Springer-Verlag, New York, 1986.
22. B. Podolsky and P. Schwed. Review of a generalized electrodynamics. *Rev. Mod. Phys.*, 20:40–50, 1948.
23. J.F. Pommaret. *Systems of Partial Differential Equations and Lie Pseudogroups.* Gordon & Breach, London, 1978.

24. J.M. Pons. Ostrogradski's theorem for higher-order singular Lagrangians. *Lett. Math. Phys.*, 17:181–189, 1989.
25. H. Rund. *The Hamilton-Jacobi Theory in the Calculus of Variations.* Van Nostrand, London, 1966.
26. D.J. Saunders. *The Geometry of Jet Bundles.* London Mathematical Society Lecture Notes Series 142. Cambridge University Press, Cambridge, 1989.
27. J. Schü, W.M. Seiler, and J. Calmet. Algorithmic methods for Lie pseudogroups. In N. Ibragimov, M. Torrisi, and A. Valenti, editors, *Proc. Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics*, pages 337–344, Acireale (Italy), 1992. Kluwer, Dordrecht 1993.
28. W.M. Seiler. *Analysis and Application of the Formal Theory of Partial Differential Equations.* PhD thesis, School of Physics and Materials, Lancaster University, 1994.
29. W.M. Seiler. On the arbitrariness of the general solution of an involutive partial differential equation. *J. Math. Phys.*, 35:486–498, 1994.
30. W.M. Seiler. Arbitrariness of the general solution and symmetries. *Acta Appl. Math.*, to appear, 1995. (Special Issue Proc. Algebraic and Geometric Structures in Differential Equations, Twente 1993, J. Krasilshik and P.H.M. Kersten, eds.).
31. W.M. Seiler. Generalized tableaux and formally well-posed initial value problems. Preprint Lancaster University, 1995.
32. P.J. Steinhardt. Problems of quantization in infinite momentum frame. *Ann. Phys.*, 128:425–447, 1980.
33. E.C.G. Sudarshan and N. Mukunda. *Classical Dynamics: A Modern Perspective.* Wiley, 1974.
34. K. Sundermeyer. *Constrained Dynamics.* Lecture Notes in Physics 169. Springer-Verlag, New York, 1982.