Involution Analysis of Field Theories*

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Abstract

We describe the application of the theory of involutive systems to field theories in mathematical physics and engineering sciences. Emphasis is put on effective constraint algorithms, on counting degrees of freedom and on the initial value problem.

1 Introduction

Field theories are of great interest in mathematical physics and in engineering sciences. On one side, we have theories for the fundamental interactions like Yang–Mills theories or general relativity. A characteristic feature of them is the presence of gauge symmetries making the field equations underdetermined. On the other side, we have more phenomenological theories like elasticity or fluid dynamics relevant in engineering applications. They often include constraints like incompressibility leading to overdetermined systems.

Two examples of field theories appear repeatedly in this article. *Maxwell's Equations* describe the evolution of the electric field $\vec{E}$ and the magnetic field $\vec{B}$ in electrodynamics:

$$\frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B}\,, \qquad \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}\,, \tag{1a}$$

$$0 = \vec{\nabla} \cdot \vec{E}\,, \qquad 0 = \vec{\nabla} \cdot \vec{B}\,. \tag{1b}$$

For simplicity, we set all constants to one and consider only the vacuum case. The *Navier–Stokes Equations* model the velocity $\vec{u}$ and the pressure p of an incompressible fluid:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla} p + \Delta \vec{u}\,, \tag{2a}$$

$$0 = \vec{\nabla} \cdot \vec{u}\,. \tag{2b}$$

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As in both cases the field equations are not normal, we cannot straightforwardly apply standard existence and uniqueness theorems. In particular, we must first check whether integrability conditions or constraints are hidden. In mathematical physics, this topic has found much interest, since Dirac has introduced in the fifties his "generalised" Hamiltonian mechanics [4]. In the engineering sciences, interest has concentrated on numerical aspects leading to so-called *(partial) differential algebraic equations* [2].

In recent years, we showed that the *formal theory of differential equations* [5, 14, 17] with its central notion of *involution* provides a powerful framework for many questions arising in this context. In a series of articles [19, 22, 24] we studied the Dirac theory; numerical questions were analysed in [11, 20, 23]. A related philosophy was used in [13, 15, 25]. In the present article, numerical questions play only a minor role. It is complementary to our earlier works emphasising an intrinsic geometric treatment; here we concentrate on effective algebraic algorithms. So we apply a broader concept of involution based on the *Janet–Riquier theory* developed around 1900 [12, 16]. Its modern algebraic formulation in terms of *involutive bases* is due to Blinkov, Gerdt and Zharkov (see [6, 7, 8, 21] and references therein); a simple introduction to the basic ideas can be found in [3].

2 Involutive Systems of Differential Equations

Our leitmotiv for introducing involution is the problem of constructing a formal power series solution for a system of differential equations. If the system is normal, this is straightforward and represents in fact the classical technique for proving the Cauchy–Kovalevskaya theorem. When we are dealing with a non-normal system, in particular with an overdetermined one, the situation is less simple. As a trivial example consider the system

$$u_x = F(u)\,, \qquad u_y = G(u) \tag{3}$$

with two yet arbitrary but fixed functions F and G. In the construction of a formal power series solution, we have two possibilities to compute the coefficient of xy: we may use the equation $u_{xy} = F'(u)u_y = F'(u)G(u)$ or the equation $u_{xy} = G'(u)u_x = G'(u)F(u)$. Obviously, the system possesses a solution, if and only if both equations produce the same result. Thus we find the *integrability condition* $F'(u)G(u) = G'(u)F(u)$ that must be satisfied by the functions F and G.

A system for which all integrability conditions are satisfied, so that we can construct a formal power series solution, is called *formally integrable* or *involutive*.[1] In the example above, one easily finds the integrability condition and in particular easily shows that it is the only condition for the formal integrability of the system. In larger examples it is more involved to recognise that one has found *all* conditions; especially, as we have not yet got a rigorous definition of what an integrability condition actually is!

[1]There exist many different approaches to this problem and so the terminology is very confusing and in particular not consistent. In the Janet–Riquier theory one traditionally speaks of *passive* systems. The notion *involutive* seems to go back to Lie and is becoming more and more popular even in Janet–Riquier and related theories where it is usually applied synonymously with *formally integrable*. As we will see below, this is not really correct. In general, involution is a stronger concept than formal integrability.

Thus we need a systematic approach for this *completion* to a formally integrable system. Unfortunately, no algorithmic criterion for formal integrability is known. The theory of *involutive bases* offers here a way out, as involution can be checked algorithmically. It is based on two fundamental ideas. The first one is the introduction of a *ranking* (or *term order*) on the set of all derivatives (the differential analogue to the terms in polynomials). It distinguishes in each equation a *leading derivative* or, for short, a *leader*; we assume in the sequel that we can solve each equation for its leader. The ranking separates the derivatives into two classes. The leaders and their derivatives are called *principal*, all remaining derivatives are *parametric*. The parametric derivatives can be chosen arbitrarily (they parametrise the formal solution space), whereas the principal derivatives are determined from the parametric ones using the differential equations.

In an arbitrary system several possibilities may exist to compute a principal derivative and they may yield different results. This observation leads to a rigorous definition of integrability conditions: they represent the difference between different ways to compute a principal derivative. In order to systematically exhibit all conditions we need to bring the system into a *normal form* based on the second fundamental idea in the theory. For each equation the independent variables are partitioned into *multiplicative* and *non-multiplicative* ones.[2] There exist different ways to do this partitioning; one speaks of an *involutive division*. A precise definition of involutive divisions can be found in [3, 7, 21].

Example 2.1. Two important divisions are the *Janet* and the *Pommaret division*. Both are based on a multi index notation for derivatives and require an ordering of the independent variables $x_1, \dots, x_n$. We write u_μ with $\mu = [\mu_1, \dots, \mu_n]$ for the derivative $\partial u / \partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}$. If u_μ is the leader of an equation, the Pommaret division assigns the multiplicative variables $x_1, \dots, x_k$ where $k = \min\{i \mid \mu_i \neq 0\}$ is the *class* of u_μ.

The definition of the Janet division is more complicated and considers all leaders of a system simultaneously. Let $\mathcal{U}$ be the set of all leaders that are derivatives of the same unknown function u. We get one such set for each unknown function and analyse each separately. We consider for $1 \leq k \leq n$ the following subsets of $\mathcal{U}$:

$$(d_k, \dots, d_n) = \left\{ u_\mu \in \mathcal{U} \mid \mu_i = d_i,\ k \leq i \leq n \right\}. \tag{4}$$

The variable x_n is multiplicative for the equation with leader u_μ, if $\mu_n = \max_{u_\nu \in \mathcal{U}}\{\nu_n\}$. The variable x_k with $k < n$ is multiplicative for an equation with leader $u_\mu \in (d_k, \dots, d_n)$, if $\mu_k = \max_{u_\nu \in (d_k, \dots, d_n)}\{\nu_k\}$. $\lhd$

Assume that we are dealing with a system of p differential equations

$$\Phi_\tau(\vec{x}, \vec{u}, \vec{u}_{\vec{x}}, \dots) = 0\,, \qquad \tau = 1, \dots, p \tag{5}$$

in n independent variables $\vec{x} = (x_1, \dots, x_n)$ and m unknown functions or dependent variables $\vec{u} = (u^1, \dots, u^m)$.[3] Let the set $\mathcal{U}_\alpha$ contain all leaders that are derivatives of u^α. If

[2] This terminology might appear strange in the context of differential equations, but Janet derived his theory in terms of monomials instead of derivatives and so used an algebraic language.

[3] We are using superscripts for labelling the dependent variables, as the subscripts are needed for the multi indices of derivatives.

the leader of the τ-th equation is $l_\tau = u^\alpha_\mu$, then let our involutive division assign the multiplicative variables $X(u^\alpha_\mu, \mathcal{U}_\alpha) = \{x_{i_1}, \dots, x_{i_k}\}$.[4] Its *involutive cone* contains all derivatives obtained by differentiating l_τ only with respect to these variables:

$$\mathcal{C}(u^\alpha_\mu, \mathcal{U}_\alpha) = \left\{u^\alpha_\nu \mid \forall i: \nu_i \geq \mu_i \text{ and } \nu_i = \mu_i \text{ if } x_i \notin X(u^\alpha_\mu, \mathcal{U}_\alpha)\right\}. \tag{6}$$

A differential equation $\Psi(\vec{x}, \vec{u}, \vec{u}_{\vec{x}}, \dots) = 0$ is in *involutive normal form* with respect to the system (5), if it does not contain any derivative u^β_ν such that $u^\beta_\nu \in \mathcal{C}(u^\beta_\mu, \mathcal{U}_\beta)$ for some leader u^β_μ. The system (5) is *involutively autoreduced*, if each of its equation is in involutive normal form with respect to the remaining ones. For linear systems it is trivial to render it involutively autoreduced. The involutive normal form of any equation can be computed algorithmically like the familiar normal forms in Gröbner bases theory. For nonlinear systems everything depends on whether we are able to solve for the leaders l_τ.

Given an involutively autoreduced system, obviously all derivatives contained in the involutive cones of the leaders are principal. Furthermore, the properties of an involutive division imply that these cones are disjoint. The question is now whether *all* principal derivatives are contained in the involutive cones of the leaders.

Definition 2.2. An involutively autoreduced system is called *involutive* (for a given ranking and involutive division), if all its principal derivatives are contained in the involutive cones of its leaders.

An alternative description is that any equation obtained by differentiating an equation of the system has zero as its involutive normal form, in other words the equation can be expressed as a linear combination of multiplicative differentiations. For linear systems this point of view leads to a differential analogue of syzygies.

Example 2.3. Consider Maxwell's Equations (1). If we order the independent variables as $\vec{x} = (x, y, z, t)$ and choose the graded reverse lexicographic ranking, then with respect to the Pommaret division the evolution equations (1a) are of class 4 while the constraints (1b) are only of class 3. Indeed all the evolution equations are solved for a time derivative which is its leader. The Gauß laws are not written in solved form but obviously we could easily solve them for their leaders, the z-derivative of the z-component of $\vec{E}$ and $\vec{B}$, respectively. The system is already involutively autoreduced.

As the evolution equations are of class 4, all variables are multiplicative for them. But the constraints are only of class 3; the time t is not multiplicative for them. So we must check whether differentiating (1b) with respect to t leads to new equations. As

$$0 = \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{E}) = \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0\,, \tag{7}$$

and similarly for $\vec{B}$, we see that this is not the case: the time derivative of (1b) can be written as a linear combination of spatial derivatives of (1a). Hence Maxwell's Equations are involutive for the Pommaret division and the graded reverse lexicographic ranking. $\lhd$

[4]Note that as in the example of the Janet division the assignment of the multiplicative variables depends in general explicitly on the whole set $\mathcal{U}_\alpha$. Only for so-called *global* divisions like the Pommaret division no such dependency occurs.

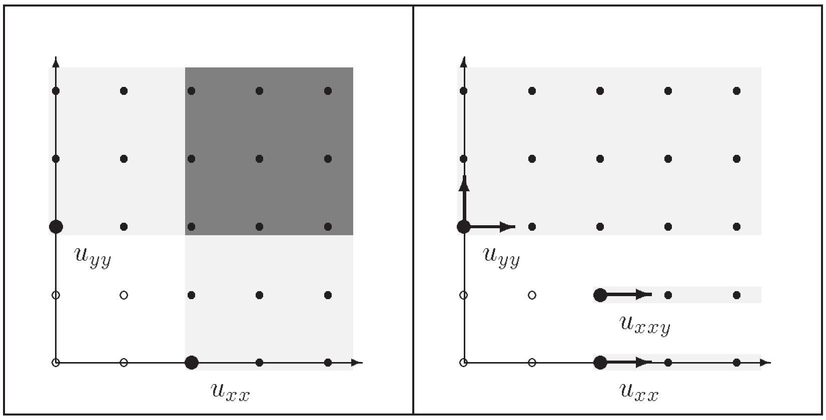

Figure 1: *Left:* principal derivatives. *Right:* involutive cones.

Example 2.4. The basic idea behind Def. 2.2 is a decomposition of the set of all principal derivatives into disjoint involutive cones. Consider the system $u_{xx} = 0$ and $u_{yy} = 0$. The principal derivatives lie in the shaded area of Fig. 1. On the left hand side one sees that many of them can be computed in two different ways, as there is an overlap between the principal derivatives determined by each of the equations. The Janet and the Pommaret division yield the same multiplicative variables and one easily checks that the system is *not* involutive, as the principal derivative u_{xxy} is not contained in the involutive cone of the leader u_{xx}. On the right hand side of Fig. 1, we added the equation $u_{xxy} = 0$ to the system which, of course, does not change the solution space. Furthermore, we marked by arrows the multiplicative variables. One clearly sees that each principal derivative lies now in precisely one involutive cone. Hence the augmented system is involutive. $\triangleleft$

Linear systems with constant coefficients are related to polynomial modules. If we assume for simplicity that there is only one unknown function, we can associate with any such system an ideal via the "Fourier transformation" $u_\mu \leftrightarrow x^\mu$. If the linear system is involutive with respect to a given ranking, then the "transformed" equations define a (non-reduced) Gröbner basis of the ideal generated by them for the same ranking.

The distinction into parametric and principal derivatives depends only on the ranking and *not* on the chosen involutive division. Given an involutive system an equation in it might have a leader who is a derivative of the leader of another equation (though of course not a multiplicative derivative). Such an equation is in a certain sense redundant; we can eliminate it and still have a formally integrable system.

This happens in Example 2.4. The original system is obviously formally integrable. The equation $u_{xxy} = 0$ is *not* an integrability condition but only an *obstruction to involution.* One may wonder why it is nevertheless useful to add this equation. One possible answer is that it becomes important in the numerical integration of the system, as obstructions to involution become integrability conditions after semi-discretisations [23]. Another answer is given in Sect. 4.

3 Completion to Involution

Most fundamental field theories are derived via variational principles. Often the analysis starts with some Lagrangian density, i. e. we are given a function $\mathcal{L}(\vec{x}, \vec{u}, \vec{u}_{\vec{x}}, \dots)$ and look for the functions $\vec{u}(\vec{x})$ minimising the action

$$S[\mathcal{L}] = \int_{\mathcal{M}} \mathcal{L}(\vec{x}, \vec{u}, \vec{u}_{\vec{x}}, \dots)\, dV\,. \tag{8}$$

Here $\vec{x}$ denotes coordinates on the space-time $\mathcal{M}$; thus this vector does not only contain spatial variables but also time. By a classical result in the calculus of variations, a necessary condition for a minimum is that it satisfies the *Euler–Lagrange equations*. If the Lagrangian density depends on derivatives up to order q, they are given by

$$\sum_{a=0}^{q}(-1)^a \sum_{|\mu|=a} \frac{\partial^a}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^a u^\alpha/\partial x^\mu)}\right) = 0\,, \qquad \alpha = 1, \dots, m\,. \tag{9}$$

In general, (9) is a system of order $2q$. Lower orders occur, if $\mathcal{L}$ depends only linearly on the derivatives of order q.

Depending on the precise form of the Lagrangian density, the system (9) is not necessarily involutive; in many cases of interest it is not in Cauchy–Kovalevskaya form and thus either under- or overdetermined. The same holds true for other field theories like the Navier–Stokes Equations. Thus we must look for hidden integrability conditions to ensure that the theory is consistent, i. e. it possesses at least formal solutions. In the physics literature one often speaks of *constraint algorithms*; in mathematical terminology this corresponds to a *completion to involution*.

Our description of involutive systems leads naturally to the simple completion algorithm sketched in Fig. 2. Its input is a system $\mathcal{H}$ of differential equations $\Phi_\tau = 0$, a ranking $\prec$ and an involutive division, i. e. a rule that assigns to each equation $\Phi_\sigma = 0$ in $\mathcal{H}$ a set $X(\Phi_\sigma, \mathcal{H})$ of multiplicative variables based on the leaders l_τ of the equations $\Phi_\tau = 0$. During the completion we always keep the system $\mathcal{H}$ in involutively autoreduced form. The algorithm proceeds by analysing all *non-multiplicative* differentiations of the equations in $\mathcal{H}$; they are collected in line /3/. We start by considering in line /5/ the one with the smallest leader with respect to the ranking $\prec$. If its involutive normal form computed in line /7/ does not vanish, it is added to $\mathcal{H}$ and we iterate. The algorithm stops, when the involutive normal form of all elements of $\mathcal{S}$ is zero.

One can show that under some rather technical conditions on the used involutive division, this algorithm always terminates with a finite involutive completion of the input system *provided that one exists*. In contrast to the situation with Gröbner bases, the existence must be proven separately. Some divisions like e. g. the Janet division are *Noetherian* meaning that *any* differential system possesses a finite involutive completion with respect to them. Thus our algorithm always terminates for the Janet division.

Example 3.1. Let us consider the Navier–Stokes Equations (2). It demonstrates a number of phenomena appearing in the completion. This is due to the fact that their evolution part is of a parabolic nature rendering the surface $t = 0$ characteristic.

Algorithm: *Completion to Involution*
Input: Finite system $\mathcal{H}$, involutive division, ranking $\prec$
Output: Equivalent involutive system

$\mathcal{H} \leftarrow$ `InvAutoReduce`$(\mathcal{H})$
`repeat`
  $\mathcal{S} \leftarrow \big\{D_x\Phi \mid \Phi \in \mathcal{H},\ x \notin X(\Phi, \mathcal{H})\big\}$
  `repeat`
    $\psi \leftarrow \min_\prec \mathcal{S}$
    $\mathcal{S} \leftarrow \mathcal{S} \setminus \{\psi\}$
    $\psi \leftarrow$ `InvNormalForm`$(\psi, \mathcal{H})$
  `until` $(\psi \neq 0)$ `or` $(\mathcal{S} = \emptyset)$
  `if` $\psi \neq 0$ `then` $\mathcal{H} \leftarrow$ `InvAutoReduce`$(\mathcal{H} \cup \{\psi\})$
  `else return` $\mathcal{H}$ `end_if`
`until` $FALSE$

Figure 2: Completion to Involution

First we attempt to complete (2) using the Pommaret division with the variable ordering $\vec{x} = (x, y, z, t)$ and the inverse lexicographic order. For the evolution equations (2a) the leaders are the time derivatives $\partial\vec{u}/\partial t$, so that all variables are multiplicative, whereas for the incompressibility condition (2b) the time t is non-multiplicative. Thus according to our algorithm we must check what happens, if we differentiate (2b) with respect to t:

$$0 = \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{u}) = \vec{\nabla} \cdot \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} \cdot \big((\vec{u} \cdot \vec{\nabla})\vec{u}\big) - \Delta p + \Delta(\vec{\nabla} \cdot \vec{u})\,. \tag{10}$$

As all spatial variables are multiplicative for (2b), the term $\Delta(\vec{\nabla} \cdot \vec{u})$ is eliminated during the determination of the involutive normal form but the other terms remain and we find as integrability condition the well-known Poisson equation for the pressure

$$\Delta p = -\vec{\nabla} \cdot \big((\vec{u} \cdot \vec{\nabla})\vec{u}\big) \tag{11}$$

playing an important role in the numerical analysis of the Navier–Stokes Equations.

This new equation is of class 3 and t is non-multiplicative for it. The corresponding differentiation yields a new equation $\Delta p_t = \dots$ with a complicated right hand side. It is again of class 3 and we must again compute its time derivative. One easily sees that this process does not stop and our algorithm does not terminate. But it is also clear that all these infinitely many equations are only obstructions to involution and not integrability conditions. Hence adding (11) to (2) yields a formally integrable system.

One can show that (after an involutive autoreduction) this augmented system is even involutive, if we change the ordering of the variables to $\vec{x} = (t, x, y, z)$. This is an example of the problem of δ-*regularity* which we cannot discuss further for lack of space. Using the Janet instead of the Pommaret division leads to a finite involutive system for any ordering of the variables. But if the "wrong" ordering is used, the completed system comprises considerably more equations and in particular equations of higher order. This is a typical effect in any completion theory, when characteristic coordinates are used. $\triangleleft$

4 Degrees of Freedom

For a general field theory it is not so obvious how one should define degrees of freedom and how one may count them. The problem becomes even more difficult in the presence of gauge symmetries. In [18, 22] we presented a solution to this problem based on the formal theory; here we rephrase it in a more algebraic language.

As in Commutative Algebra one can introduce a *Hilbert function* for differential equations: $h(r)$ counts the number of parametric coefficients of order r. Thus it measures the size of the formal solution space of the system. Already Janet gave closed form expressions for the Hilbert function of an *involutive* system. In our language we can write it for an involutive system $\Phi_\tau = 0$ with $1 \leq \tau \leq p$ as follows [6]

$$h(r) = m\binom{n+r}{r} - \sum_{\alpha=1}^{m}\sum_{j=1}^{r}\sum_{\tau=1}^{p}\binom{j - \mathrm{ord}(l_\tau) + \mathrm{cls}(l_\tau) - 1}{\mathrm{cls}(l_\tau) - 1} \tag{12}$$

where again l_τ is the leader of the equation $\Phi_\tau = 0$, $\mathrm{ord}(l_\tau)$ denotes its order and $\mathrm{cls}(l_\tau)$ gives the number of its multiplicative variables. m and n are the number of dependent and independent variables, respectively. (12) is a simple consequence of disjoint decomposition of the set of all principal derivatives induced by an involutive system. Indeed, the first summand gives the total number of derivatives of order r; then we subtract the number of principal derivatives of this order which can be easily computed via the decomposition. Thus a merely formally integrable system would not suffice here.

The Hilbert function becomes polynomial for r large enough. In our case, it is easy to give an upper bound for the order where this is happening: the maximal order q of an equation in our involutive system. The *Hilbert polynomial* is explicitly given by

$$H(r) = m\binom{n+r}{r} - \sum_{\alpha=1}^{m}\sum_{\tau=1}^{p}\binom{r - \mathrm{ord}(l_\tau) + \mathrm{cls}(l_\tau)}{\mathrm{cls}(l_\tau)}\,. \tag{13}$$

Any numerical polynomial can be expressed as a linear combination of binomial coefficients. For the Hilbert polynomial the following form is convenient:

$$H(r) = \sum_{i=1}^{n}\alpha_q^{(i)}\binom{r-q+i-1}{i-1}\,. \tag{14}$$

The coefficients $\alpha_q^{(i)}$ are called *Cartan characters*; they can be related to the number of arbitrary functions in the general solution of the system [18].

Field theories typically describe evolution problems. A distinguished independent variable *time* t exists and one studies initial value problems (see Sect. 5): an initial state is prescribed and its evolution analysed. The number of *degrees of freedom* corresponds to how much data one needs to describe uniquely a state. Whereas in finite-dimensional theories this corresponds to a finite number of parameters, initial data for field theories are functions. Thus it is not so clear how one should count degrees of freedom.

In [22] the following approach was proposed. We start by considering normal field equations. For a system in Cauchy–Kovalevskaya form one may prescribe for each function its value at $t = 0$; thus the initial data are m functions defined on an $(n-1)$-dimensional Cauchy surface and intuitively we would consider this as a system with m degrees of freedom. We generalise this definition to arbitrary involutive system by defining the number of degrees of freedom as the number of functions in the initial data that can take arbitrary values on an $(n-1)$-dimensional Cauchy surface; functions of less arguments are not taken into account.

Assuming for the moment that our field equations are not underdetermined (this is equivalent to $\alpha_q^{(n)} = 0$), one can show that this number is the Cartan character $\alpha_q^{(n-1)}$. Thus the Hilbert polynomial allows us straightforwardly to determine the number of degrees of freedom. A somewhat similar approach has been proposed by Einstein (cf. [18]).

Example 4.1. The Hilbert polynomial of Maxwell's Equations is $H(r) = 2r^2 + 8r + 6$ and thus the Cartan characters are $\alpha_1^{(4)} = 0$, $\alpha_1^{(3)} = 4$ and $\alpha_1^{(2)} = \alpha_1^{(1)} = 6$ implying that electromagnetism has four degrees of freedom. $\lhd$

Now we must include underdetermined theories. From a physical point of view such an underdeterminacy may only stem from a *gauge symmetry*, otherwise the theory is not a reasonable model. Thus there must exist a group of transformations of the form

$$\vec{u}' = \vec{T}[\vec{\Lambda}(\vec{x})](\vec{u}) \tag{15}$$

where $\vec{\Lambda}(\vec{x})$ is a vector (of some dimension) of arbitrary functions; the bracket indicates that $\vec{T}$ may also depend on derivatives of $\vec{\Lambda}$ up to some finite order. The group is a gauge group, if the action (8) remains invariant under these transformations. This implies in particular that the transformations map solutions of the field equations again into solutions: if $\vec{u}(\vec{x})$ is a solution of the field equation, applying to it the transformation (15) yields a new solution $\vec{u}'(\vec{x})$ for any value of the gauge functions $\vec{\Lambda}(\vec{x})$.

Gauge transformations cannot be observed in experiments, hence we identify solutions related by such a transformation: they correspond to the same physical state. This affects the degrees of freedom count; we do not measure the full formal solution space but only its quotient by the gauge group. Under some technical assumptions it is possible to derive expressions for the resulting correction to the Cartan characters [18], but this is rather tedious. A much more elegant method consists of using *Lie pseudogroups*.

In this approach one does not consider the transformations in their explicit form (15) but instead a system of differential equations whose solutions are these transformations. Then we can perform an involution analysis of this system and in particular determine its Hilbert polynomial $G(r)$. The *gauge corrected Hilbert polynomial* is simply the difference

$\bar{H}(r) = H(r) - G(r)$ and one can proceed as above to determine the *gauge corrected Cartan characters* $\bar{\alpha}_q^{(k)}$. If the gauge group was properly identified so that indeed all underdeterminacy stems from its action, we find that $\bar{\alpha}_q^{(n)} = 0$ and the number of physical degrees of freedom of the gauge theory is given by $\bar{\alpha}_q^{(n-1)}$.

Example 4.2. Electromagnetism may also be described as the $U(1)$ Yang–Mills theory. Here the unknown functions are four fields A_μ (the vector potential) with $0 \leq \mu \leq 3$ and the Euler-Lagrange equations are the following second order system

$$\sum_{\mu=0}^{3} \left(\frac{\partial^2 A_\nu}{\partial x_\mu^2} - \frac{\partial^2 A_\mu}{\partial x_\mu \partial x_\nu} \right) = 0\,, \qquad \nu = 0, \ldots, 3\,. \tag{16}$$

Here we denote by x_0 the time and by x_1, x_2, x_3 the spatial variables. The relation to Maxwell's Equations is given by

$$\vec{E} = \vec{A}_t - \vec{\nabla} A_0\,, \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{17}$$

where the vector $\vec{A}$ consists of A_1, A_2, A_3. One readily checks that (16) remains invariant under transformations $A'_\mu = A_\mu + \partial_{x_\mu}\Lambda$ with an arbitrary function Λ. Such transformations also do not affect the definition of the electric and magnetic field in (17), so that we may consider Maxwell's Equations (1) as a gauge invariant form of (16).

(16) is involutive for both the Janet and the Pommaret division. Its Hilbert polynomial is $H(r) = \frac{1}{6}r^3 + \frac{7}{2}r^2 + \frac{25}{3}r + 4$ which yields the Cartan characters $\alpha_2^{(4)} = 1$ (corresponding to the one gauge function Λ), $\alpha_2^{(3)} = 7$, $\alpha_2^{(2)} = 12$ and $\alpha_2^{(1)} = 16$. In the pseudogroup approach we describe our gauge transformations as the solutions of the linear first order system

$$\frac{\partial A'_\mu}{\partial x_\nu} - \frac{\partial A'_\nu}{\partial x_\mu} = 0\,, \qquad \frac{\partial A'_\mu}{\partial A_\nu} = \delta_\mu^\nu\,, \qquad\qquad \mu, \nu = 0, \ldots, 3\,. \tag{18}$$

It is already involutive for our divisions and its Hilbert polynomial is readily determined to $G(r) = \frac{1}{6}r^3 + \frac{3}{2}r^2 + \frac{13}{3}r + 4$. So the gauge corrected Hilbert polynomial is $\bar{H}(r) = 2r^2 + 4r$ leading to the Cartan characters $\bar{\alpha}_2^{(4)} = 0$, $\bar{\alpha}_2^{(3)} = 4$ and $\bar{\alpha}_2^{(2)} = \bar{\alpha}_2^{(1)} = 6$. Thus after the gauge correction we find the same number of degrees of freedom as in Maxwell's theory. $\bar{H}(r)$ is not identical with the Hilbert polynomial of Maxwell's Equations: (16) is of second order, (1) of first order; so the Hilbert polynomial of Maxwell's Equations is given by $\bar{H}(r+1)$ (cf. [22]). $\lhd$

5 Initial Value Problems

The results in the previous section concern only formal solutions, as they are based on counting the parametric derivatives. In particular, they do not imply the existence of a strong solution. Provided we are using a *graded* or *orderly* ranking where derivatives of higher order are always greater than derivatives of lower order, *Riquier's Theorem* guarantees the existence of analytic solutions for involutive analytic systems. The restriction to graded rankings is essential and cannot be relaxed, as the following simple example demonstrates.

Example 5.1. Consider the *heat equation* $u_t = u_{xx}$. Assume that we are using a ranking such that u_t is the leader, so that we are considering the initial value problem $u(x,0) = f(x)$. It is well-known that the solution of this problem can be written as

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4t}\right] f(\xi)\, d\xi\,. \tag{19}$$

Obviously, this integral does not exist for arbitrary analytic initial data $f(x)$ but we must impose some conditions on its behaviour for $x \to \pm\infty$. $\lhd$

For normal systems the formulation of an initial value problem is straightforward: choose a non-characteristic surface and prescribe values for all unknown functions on this surface. For an overdetermined system we could in principle do the same, but the initial data could not be chosen arbitrarily but must satisfy compatibility conditions. Thus solutions would exist only for particular choices of the initial data. Conversely, for an underdetermined system the initial data would not suffice to determine a unique solution.

We call an initial value problem *formally well-posed*, if it possesses for arbitrary initial data (consisting of formal power series) a unique formal power series solution. Note that compared with the traditional notion of well-posedness we exclude the continuous dependency on the initial data. The theory of involutive bases offers us a systematic way to construct formally well-posed initial value problems. Obviously, the coefficients of the power series given as initial data must be in a one-to-one correspondence to the parametric coefficients of the given involutive system in order to achieve formal well-posedness.

The idea is to proceed as in the definition of an involutive system: we decompose the set of all *parametric* derivatives into disjoint cones. If $\mathcal{P}_\alpha$ denotes the set of all parametric derivatives of u^α and $\mathcal{U}_\alpha$ correspondingly for the principal derivatives, then for any Noetherian involutive division the set $\mathcal{P}_\alpha$ has a disjoint decomposition

$$\mathcal{P}_\alpha = \bigcup_{g\in\mathcal{G}_\alpha} \bar{\mathcal{C}}(g) \tag{20}$$

where $\mathcal{G}_\alpha \subseteq \mathcal{P}_\alpha$ is a finite set and $\bar{\mathcal{C}}(g) \subseteq \mathcal{C}(g, \mathcal{U}_\alpha \cup \{g\})$ [6]. The last condition says that $\bar{\mathcal{C}}(g)$ is a subcone of the involutive cone $\mathcal{C}(g, \mathcal{U}_\alpha \cup \{g\})$ determined according to (6).

The decomposition (20) is not uniquely defined. For the Janet division one can show that the set $\mathcal{G}$ can always be chosen in such a way that $\bar{\mathcal{C}}(g) = \mathcal{C}(g, \mathcal{U}_\alpha \cup \{g\})$ for all $g \in \mathcal{G}$. As the Pommaret division is not Noetherian, the above result does not apply to it. But one easily sees that in this case a decomposition of the form

$$\mathcal{P}_\alpha = \mathcal{G}_0 \cup \bigcup_{g\in\mathcal{G}_1} \mathcal{C}(g) \tag{21}$$

exists where $\mathcal{G}_0 = \{g \in \mathcal{P}_\alpha \mid \mathrm{ord}(g) < q\}$ and $\mathcal{G}_1 = \{g \in \mathcal{P}_\alpha \mid \mathrm{ord}(g) = q\}$ with q the order of our system of differential equations. Here $\mathcal{C}(g)$ is the involutive cone of the derivative g for the Pommaret division (recall it is global) and the number of elements of $\mathcal{G}_1$ of class k is given by the Cartan character $\alpha_q^{(k)}$.

Given a disjoint decomposition of the form (20) for each set $\mathcal{P}_\alpha$ we associate with it the following initial value problem for our system. Let $g = u^\alpha_\mu \in \mathcal{G}_\alpha$ and assume that the cone $\bar{\mathcal{C}}(g)$ is spanned by the multiplicative variables $x_{i_1}, \dots, x_{i_{k_g}}$. Then we prescribe for the derivative u^α_μ an arbitrary function of these variables setting the remaining variables to a constant value, say, zero. Thus we obtain one initial condition for each generator g of the decomposition. By construction, this yields a formally well-posed initial value problem, as each parametric derivative in the problem is uniquely determined. Riquier's Theorem asserts that for a graded ranking the arising formal power series solution converges to an analytic solution provided analytic functions are used as initial data.

Example 5.2. We construct a formally well-posed initial value problem for Maxwell's Equations. We saw already above that they are involutive for the Pommaret division and the graded reverse lexicographic ordering. The initial conditions are easily seen to be

$$\begin{aligned} E^x(x,y,z,0) &= f_1(x,y,z)\,, & B^x(x,y,z,0) &= g_1(x,y,z)\,,\\ E^y(x,y,z,0) &= f_2(x,y,z)\,, & B^y(x,y,z,0) &= g_2(x,y,z)\,,\\ E^z(x,y,0,0) &= f_3(x,y)\,, & B^z(x,y,0,0) &= g_3(x,y)\,. \end{aligned} \tag{22}$$

Note that we prescribe values for the z-components of $\vec{E}$ and $\vec{B}$ only on a two-dimensional surface. By Riquier's Theorem there exists a unique analytic solution for the initial value problem (1,22) provided the functions f_i, g_i are analytic. Thus the general solution indeed depends only on four functions of three variables, as predicted by our degrees of freedom count in the last section. $\triangleleft$

This example also demonstrates a weakness of our formal approach. The constraints in Maxwell's Equations form an elliptic subsystem. It is well-known that for elliptic systems initial value problem are instable, as the solution does not depend continuously on the initial data; arbitrarily small perturbations have a large effect on the solution. Furthermore, one must clearly say that the restriction to analytic equations and initial data is not acceptable in applications. [23] contains an existence and uniqueness theorem for smooth solutions of a class of systems including Maxwell's Equations. But such theorems can only be proven under additional assumptions on the structure of the system.

6 Conclusions

We gave an overview how field theories benefit from an involution analysis. For lack of space we used a rather informal style; technical details are found in the cited references. For a physicist the problems we discussed may not be the most important ones. But one should stress that almost any question concerning a system of differential equations can be answered reasonably only *after* a completion to involution. This begins already with the classification into hyperbolic or elliptic systems making sense only for involutive systems. Thus the involution analysis is a necessary first step for the deeper study of any field theory.

We omitted a discussion of implementation and efficiency issues of the completion algorithm. This is an active field of research. Many optimisations of the simple algorithm presented here are possible; [9] discusses some and also contains comparisons of the involutive approach to Gröbner bases with the classical one for classical benchmark problems.

This article used an algebraic and thus necessarily coordinate dependent approach to the completion. In modern physics intrinsic geometric formulations are preferred and the formal theory offers an intrinsic completion method. We combined recently both theories into a single hybrid algorithm [10]. On the algebraic side the algorithm produces an involutive system for the Pommaret division and the graded reverse lexicographic ranking; on the geometric side it allows us to recover every step of the intrinsic Cartan–Kuranishi completion of the formal theory. Its correctness is proved only for linear systems. Our implementation in *MuPAD* (a preliminary description may be found in [1]) can also treat systems with polynomial non-linearities in the derivatives. But one must be careful with the interpretation of the results, if pseudo-reductions had to be applied, as one might leave the differential ideal generated by the system. Nevertheless, one can for example complete without problems the Navier–Stokes equations.

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