COMPUTER ALGEBRA AND FIELD THEORIES

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ABSTRACT. We discuss some non-standard applications of computer algebra in field theories based on the formal theory of differential equations and present a general programming environment for differential equations currently under construction in MuPAD.

1. Introduction

The application of computer algebra to field theories has so far been dominated to a large extent by high energy physicists computing Feynman diagrams. In fact, this application has been quite important for the history of computer algebra, as it was one of the driving forces behind the development of the first general purpose systems like REDUCE in the sixties.

Today, thirty years later, the generation and computation of Feynman diagrams still represents the main application of computer algebra in field theories and has become a huge "industry". Other topics have remained relatively exotic. There exist some packages for tensor manipulations (though they are more popular in general relativity) and some packages for calculations with Lie algebras or super fields, see e.g. [3, 8, 12, 16]. However, none of them has found wide spread use.

Many high energy physicists have forgotten that the core of a field theory consists of a system of partial differential equations, namely the field equations. This is in marked contrast to other domains like elasticity or fluid dynamics where the analysis of the field equations lies in the very center of the theory.

It is the purpose of this short note to show how the formal theory of differential equations provides not only a nice theoretical framework to study many important issues in field theory but also allows us to approach them in an algorithmic way well adapted to computer algebra. The theoretical aspects will be covered in the next section. Section 3 discusses the implementation in a computer algebra system. Finally, some conclusions are given.

2. Field Theories and Differential Equations

Most modern field theories are gauge theories. At the level of the field equations this entails that these do not any longer form a normal system (or system in Cauchy-Kowalevsky form). In a normal system there exists a distinguished independent variable (usually the time t) such that every equation can be solved for a t-derivative and on the right hand sides no t-derivatives occur. Due to the presence of the gauge symmetry, field equations are under-determined and constraints appear.

1991 *Mathematics Subject Classification.* Primary 68Q40, 70G50, Secondary 35G20.
Key words and phrases. Field theories, differential equations, computer algebra.
This work was supported by Deutsche Forschungsgemeinschaft.

The treatment of systems with constraints was pioneered by Dirac [5, 6]. But he also introduced the somewhat dubious “strategy” that has been taken up by most of the literature on this subject: theorems are proven only for finite-dimensional systems (i.e. for ordinary differential equations) but nevertheless applied without hesitation to field theories. We pointed out some caveats and problems of this approach in two recent articles [21, 23].

In the first [23] of these articles we showed that from a mathematical point of view the Dirac theory is just a special case of the general problem of completing a system of differential equations. Even for systems of partial differential equation general solutions of this problem have been developed long before Dirac. However, they remained fairly unknown. Two examples are the Janet-Riquier theory [11] with the notion of a passive system and the Cartan-Kähler theory [2] of exterior systems with the notion of involution.

We used in our above mentioned works on constrained dynamics the formal theory of differential equations [13, 17]. It combines elements of Janet-Riquier and Cartan-Kähler theory and provides a general intrinsic completion algorithm, the Cartan-Kuranishi algorithm. For finite-dimensional Hamiltonian systems it coincides with the Dirac algorithm; in the infinite-dimensional case we could show that in general the Dirac algorithm does not suffice to prove the consistency of a field theory, as it does not always complete the field equations.

Another important aspect of constrained dynamics besides the question of completion is to count the degrees of freedom of a theory. In [23] we presented a new intrinsic approach to this question. It is based on earlier work [18, 20] on the arbitrariness of the general solution of involutive systems with special emphasis on the correct treatment of gauge symmetries. As a concrete example we studied in some detail the two-dimensional dilaton gravity [24].

3. Computer Algebra

Many applications of computer algebra to differential equations have been reported in the literature. However, most of them are stand-alone packages, i.e. the author(s) developed an algorithm for some specific problem and implemented it. Since no computer algebra system provides default data structures for differential equations, every author designs his own structures.

For simple tasks this approach may suffice, but for more complex problems it faces limitations. Imagine that you want to apply the Lie symmetry theory to some system of partial differential equations. You will probably use one of the many symmetry packages [10] to set up the determining equations. In order to solve these it is often useful to complete them to a passive or involutive system. Again you can choose from a number of packages, e.g. [14]. Finally, you may have found a symmetry and want to perform a corresponding change of variables. Procedures for this are contained in various packages, e.g. [4].

So far this sounds promising. But in practice you will discover that the packages have been written by different authors and are based on different data structures. Thus you must write conversion routines in order to be able to process the output of one package with another one. This is not only time consuming, often it is almost impossible, as the used data structures are not always well documented.

One way out is a toolbox like [4] with many utility procedures for differential equations. But this provides only a partial solution. Complex algorithms like the Cartan-Kuranishi completion are based on subalgorithms like determining the dimension of a differential equation. The implementation of these might differ for different classes of differential equations: for linear systems Gaussian elimination suffice; for a polynomial system one must apply much more expensive Gröbner bases techniques.

In conventional computer algebra systems no really satisfying solution for this problem exists. Essentially, one must implement a separate completion procedure for each class of differential equations which makes maintenance quite tedious. In principle, computer science has already developed an answer for this and similar problems: object-oriented programming with abstract data types.

Only few computer algebra systems apply this modern approach. AXIOM is completed based on it. MuPAD has the `domains` library [7] which is closely integrated into the system. The `domains` library of MAPLE (formerly called GAUSS [9]) is on the surface similar to the one in MuPAD but much less integrated into the system. However, one must say that object-oriented techniques in computer algebra still belong to the "non-standard applications".

We have started developing an object-oriented programming environment for differential equations. A first prototype was implemented in AXIOM and proved to be very useful in the development of a completion and a symmetry package. Details and examples of its use can be found in [15, 19]. Currently we are porting the environment to MuPAD where it will be part of future versions of the system.[1]

4. Conclusions

We briefly discussed a novel mathematical approach to field theories based on the formal theory of partial differential equations and the application of computer algebra to it. This new point of view directly relates a physical theory with a well established mathematical theory. Whereas the Dirac algorithm solves one special problem in constrained dynamics, the problem of completion is much more general and occurs in many fields. Thus our completion package is not only more powerful than previous implementations [25] of the Dirac theory, it can also be applied to a much wider range of problems.

Another "rediscovery" of the completion problem is currently happening in numerical analysis. Here it appears in the form of the (differential) index [1] of a differential algebraic equation. Again the theory is rather well understood in the finite-dimensional case, but the extension to infinite-dimensional system proves to be rather difficult. A brief discussion of the application of formal theory to this domain and its relation to the Dirac theory can be found in [22].

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