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Index Concepts for General Systems of Partial Differential Equations

We proposed recently some index concepts for general systems of partial differential equations [12] based on the so-called formal theory. In this note we discuss their meaning in the context of semi-discretisations. It is shown for some simple examples that the involution index of the original partial differential equations provides a lower bound for the index of the differential algebraic equations obtained by a semi-discretisation.

1. Involutive Systems of Differential Equations

Differential algebraic equations have now been studied intensively for more than 20 years [1]. However, the extension of this theory to partial differential equations has only just begun and encounters many difficulties. We went recently the opposite direction and demonstrated that the formal theory of differential equations provides a powerful framework for studying general systems of partial differential equations, i.e. systems that do not necessarily satisfy the conditions of the Cauchy-Kowalevsky theorem, and that its specialisation to ordinary differential equations recovers much of the standard index theory of differential algebraic equations [12].

The notion of an *involutive system* of differential equations is central for the formal theory. It is beyond the scope of this short note to give a rigorous definition of it. For this we must refer to the literature [8,11]. In this section we will try to explain, in as simple words as possible, some of the basic ideas behind involution. We will restrict to linear systems, although from a theoretical point of view there is no fundamental difference between linear and non-linear systems in this context. The first step towards an involutive system of differential equations is *formal integrability*. The name stems from the fact that for formally integrable systems it is straightforward to construct order by order formal power series solutions. This will be the case, iff there are no *integrability conditions* hidden in the system. Such a condition represents a differential equation that is automatically satisfied by any solution of the given system but that is nevertheless algebraically independent of it.

For ordinary differential equations only one mechanism for the generation of integrability conditions exists: the system contains equations of differing order and the differentiation of the lower order ones leads to new equations. A simple example is $\dot{u} = v$ and $u + v = 0$. Differentiating the second equation and substituting the first one yields $\dot{v} = -v$. As for first order systems lower order equations are in fact algebraic equations, the name *differential algebraic equations* has become costumery for such systems.

For partial differential equations a second mechanism exists consisting in the simplest case of cross-differentiations. Consider the system $u_{zz} = u_y$ and $u_{zy} = u_x$. By subtracting the y-derivative of the first equation from the z-derivative of the second one, we see that any solution of this system must also satisfy the equation $u_{yy} = u_{xz}$. This equation cannot be constructed by performing only algebraic manipulations of the original system; it is crucial that we differentiate. More generally, integrability conditions arise, if it is possible to take linear combinations of differentiated equations such that all derivatives of highest order cancel. If we cannot generate any new equation this way, no matter how often we differentiate, then our system is formally integrable. One can show that in our example above addition of the integrability condition yields a formally integrable system.

A formally integrable system is called *involutive*, if it possesses in addition a somewhat abstract property of a more combinatorial nature. This property sounds very bizarre at first sight, but there rests a surprising power in it which explains much of the importance of involution. As we assume that we are dealing with a linear system, we can solve each equation in it for a derivative of maximal order. With some algebraic manipulations we can achieve that each equation is solved for a different derivative and thus we may consider each equation as a unique rule to determine a certain derivative. But when we start to differentiate the equations, we loose in general this uniqueness. In our example above, the integrability condition arises from the fact that there are two possibilities to determine u_{zzy} and we must of course require that both yield the same result.

Assume now that, somehow, we knew already that the given system was formally integrable, so that, whenever there exist several possibilities to compute a derivative, all of them lead to the same result (modulo the equations contained in the system). Involution concerns a certain "recipe" to choose in such cases a unique way to determine each derivative. It requires an ordering of the independent variables; so let us call them $x_1, \dots, x_n$. If $u_{x_{i_1} x_{i_2} \cdots x_{i_q}}$ is a derivative of order q with $i_1 \leq i_2 \leq \cdots \leq i_q$, we call i_1 the *class* of the derivative.

If an equation in our system is solved for a derivative of class k, we differentiate it only with respect to the variables $x_1, \ldots, x_k$. It is not so difficult to see that if one has chosen carefully the derivatives for which each equation is solved (this means basically that one has taken derivatives of as high class as possible), all equations one obtains by these differentiations are algebraically independent, as they are still all solved for different derivatives. The decisive question is now whether we can construct *all* possible differentiated equations this way or whether we perhaps miss some. If we indeed get all equations, we call the system involutive.

We demonstrate this idea with two simple examples. Consider first the system $u_{x_2x_2} = 0$ and $u_{x_1x_2} = 0$. The first equation is of class 2 and we allow all possible differentiations of it. The second equation is of class 1 and thus may be differentiated only with respect to x_1. But this restriction has no consequences: the equation $u_{x_1x_2x_2} = 0$ can also be obtained by an "allowed" differentiation of the first equation with respect to x_1. Thus we have an involutive system. Now we take the system $u_{x_2x_2} = 0$ and $u_{x_1x_1} = 0$. Again the first equation is of class 2 and the second of class 1. But now the restriction to "allowed" differentiations prohibits us to derive the equation $u_{x_1x_1x_2} = 0$. So we have here an example of a system that is not involutive.

2. Completion and Indices

Involutive systems share a number of useful properties. For example, in the case of analytic systems one can prove an existence and uniqueness theorem for analytic solutions of the Cauchy problem, the *Cartan-Kähler theorem*. Involutive systems may also be considered as a kind of generalisation of the well-known Gröbner bases of polynomial ideals [2] to differential equations. In fact, for linear systems this relation can be made precise. Reinterpreting any derivative $u_{xxy\cdots}$ as a monomial $x^2y\cdots$, we can associate to every linear system with constant coefficients a set of polynomials. If the system is involutive, this set represents not only a Gröbner basis of the ideal generated by it but even a very special kind, a so-called involutive basis, possessing many interesting properties [3,13].

If a given system of differential equations is not involutive, the *Cartan-Kuranishi theorem* asserts that we can make it involutive by differentiating it finitely often and by adding a finite number of integrability conditions. The proof of this theorem even leads straightforwardly to an algorithm for this *completion to involution*. An efficient version of it and its implementation in the computer algebra system MuPAD will be described in two forthcoming publications [5,6]. For lack of space we cannot discuss this interesting problem any further.

The basic insight of [12] was that differentiation indices, as they are often used to classify differential algebraic equations, basically only count the number of steps this completion algorithm requires; we thus prefer to speak of the *involution index*. As the Cartan-Kuranishi theorem holds for ordinary and for partial differential equations, we obtain this way automatically an index concept that can be applied independently of the number of variables. One can show that for linear differential algebraic equations it coincides with the *strangeness index*[1] introduced in [7].

For ordinary differential equations the theory becomes rather trivial, as for them there is no difference between involution and formal integrability and one can omit all the complicated combinatorial considerations. So it is not surprising that the completion algorithm restricted to this case has been rediscovered many cases, see e.g. [9,10]. For Hamiltonian systems with constraints one recovers the classical Dirac theory [14].

It should be emphasised that we speak here exclusively about *differentiation indices* and not about *perturbation indices* as introduced by Hairer et al. [4]. For the latter class of indices it is still an open question how they can be formulated for partial differential equations. Involution should play here an important role, too, as the involution index gives a sharp upper bound for the order of the derivatives of the perturbations that enter the involutive system and thus its solution.

3. Semi-Discretisations

Much of the currently existing literature on designing indices for general systems of partial differential equations is based on semi-discretisations, i.e. one reduces to a differential algebraic system for which many index concepts exist. We will present now two examples as evidence for a conjecture: *the involution index of a system of partial differential equations provides a lower bound for the index of any differential algebraic system arising from it by a semi-discretisation.*

In a first simple example we demonstrate the effect of the existence of integrability conditions in a system of partial differential equation on its semi-discretisation. Not very surprising, one rediscovers discretised versions of the integrability conditions as "hidden" constraints of the arising differential algebraic system. Consider the following

[1]Due to a slightly different way of counting the strangeness index is actually always 1 less than the involution index.

linear over-determined system of second order partial differential equations for an unknown function $u = u(t, x)$:

$$u_{tt} = u_x \,, \quad u_{xt} = 0 \,. \tag{1}$$

This system is not formally integrable, as a trivial cross differentiation yields the integrability condition $u_{xx} = 0$. Addition of this equation yields an involutive system, so we can conclude that the involution index of (1) is 1.

Now we perform a semi-discretisation of (1) by substituting central differences for the x-derivatives. We omit a discussion of initial or boundary conditions — not only for simplicity but also because for a non-involutive system it is not so obvious what conditions can be imposed. In our example, it is easy to see that the general solution of system (1) is given by $u(t, x) = 2Ax + At^2 + Bt + C$ with three arbitrary real constants A, B and C. Thus only three conditions at *points* could be imposed. The semi-discretisation yields the following differential algebraic system

$$2h\ddot{u}_i = u_{i+1} - u_{i-1} \,, \quad \dot{u}_{i+1} - \dot{u}_{i-1} = 0 \tag{2}$$

where h denotes the step width of the x-discretisation and $u_i = u_i(t)$ approximates the values of $u(t, x)$ at the i-th grid point. In order to determine the differentiation index of (2), we must check what happens, if we differentiate the second equation. After some trivial manipulations we get the algebraic equation

$$u_{i+2} - 2u_i + u_{i-2} = 0 \tag{3}$$

which represents just a central difference approximation (with grid size $2h$) of the integrability condition $u_{xx} = 0$ of the original system (1). It is easy to check that differentiation of (3) does not yield any new equations, so that the index of the differential algebraic system (2) is 1 and thus equal to the involution index of (1).

One should note that this result does not depend on the fact that we used central differences. It is easy to see that no matter what kind of difference formula is used to approximate the x-derivatives, one will always obtain a differential algebraic system consisting of second and first order equations and the differentiation of the first order equations will yield some algebraic constraints. Thus the differentiation index is always at least 1.

While this result is no big surprise, it is perhaps less obvious that when one discretises a formally integrable but not involutive system of partial differential equations the arising differential algebraic system will contain hidden constraints, too. As a simple example for this effect, we consider again a linear over-determined system of second order partial differential equations for an unknown function $u = u(t, x)$.

$$u_{tt} = 0 \,, \quad u_{xx} = 0 \,. \tag{4}$$

This system is formally integrable, as it obviously cannot generate any integrability conditions. But it is *not* involutive; it becomes involutive only after one differentiation to third order. So we find an involution index of 1.

As above we perform a semi-discretisation by approximating the x-derivative by a central difference and obtain the differential algebraic system

$$\ddot{u}_i = 0 \,, \quad u_{i+1} - 2u_i + u_{i-1} = 0 \,. \tag{5}$$

Obviously, we get new first order equations by differentiating the algebraic equations:

$$\dot{u}_{i+1} - 2\dot{u}_i + \dot{u}_{i-1} = 0 \,. \tag{6}$$

It is easy to check that differentiating (6) does not yield any new equations and thus the index of the differential algebraic system (5) is 1, too.

Finally, let us consider the differentiation of (4) to third order. It yields the following *involutive* system

$$u_{ttt} = 0 \,, \quad u_{ttx} = 0 \,, \quad u_{txx} = 0 \,, \quad u_{xxx} = 0 \,. \tag{7}$$

Approximating the x-derivatives by central differences leads to the differential algebraic system

$$u_i^{(3)} = 0 \,, \quad \ddot{u}_{i+1} - \ddot{u}_{i-1} = 0 \,, \quad \dot{u}_{i+1} - 2\dot{u}_i + \dot{u}_{i-1} = 0 \,, \quad u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2} = 0 \,. \tag{8}$$

It is left as an exercise for the reader that differentiation of any of the lower order equations in (8) does not yield anything new. Thus there are no hidden constraints and both (7) and (8) are of index 0.

4. Conclusions

We have outlined some ideas from the formal theory of differential equations and how they are related to index concepts for differential algebraic equations. Based on two simple examples we have formulated a conjecture on the behaviour of the index under semi-discretisation. Attempts to prove this conjecture lead to interesting questions in Commutative Algebra related to *syzygies.*

All examples in this note are completely artifical and have no relevance for concrete applications. But there exist very important systems of partial differential equations which are not in Cauchy-Kowalevsky form. *Maxwell's equations* governing electromagnetic fields

$$\partial_t \vec{E} = \mathrm{rot}\,\vec{B}\,, \quad \partial_t \vec{B} = -\mathrm{rot}\,\vec{E}\,, \quad \mathrm{div}\,\vec{E} = 0\,, \quad \mathrm{div}\,\vec{B} = 0 \tag{9}$$

form an involutive system, although they are over-determined. The *incompressible Navier-Stokes equations*

$$\partial_t u + (u \cdot \nabla)\, u = \,-\,\mathrm{div}\, p + \Delta\, u\,, \quad \mathrm{div}\, u = 0 \tag{10}$$

have an involution index of 2 because of the peculiar way the pressure p enters. There is one non-trivial integrability condition, namely the well-known Poisson equation for the pressure

$$\Delta p + \nabla \cdot (u \cdot \nabla)\, u = 0\,. \tag{11}$$

It has been shown in [15] that any semi-discretisation of the Navier-Stokes equations indeed leads to a differential algebraic equation with a strangeness index of at least 1 and thus an involution index of at least 2.

In general, one expects that a semi-discretisation leads to a differential algebraic system with an index equal to the one of the original system of partial differential equations. In fact, one should take care that this is the case. The index is an important structural property of a differential system. If it is changed by the semi-discretisation, one must expect that other properties of the system may be changed, too, and so the obtained numerical approximation might be not very reliable.

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