.1

Marcus Hausdorf, Werner M. Seiler

Perturbation versus Differentiation Indices

We discuss the relation between differentiation and perturbation indices of over-determined systems of differential equations. It is shown that the determinacy index of the perturbed system essentially gives the perturbation index of the original system. We indicate how this result can be extended to systems of partial differential equations.

1. Introduction

Indices are of great importance for the numerical treatment of differential algebraic equations [1]. One may distinguish two main classes: *differentiation* and *perturbation indices*. The former ones count the number of differentiations needed until the system satisfies some conditions; the latter ones are based on estimates for a perturbed equation. The precise relation between the two classes was an open question for a rather long time, until Campbell and Gear obtained some definite results with the careful introduction of appropriate differentiation indices [2].

In recent times, interest has grown in generalising the theory of differential algebraic equations to non-normal systems of partial differential equations, i.e. to systems not in Cauchy–Kovalevskaya form. In an earlier article [7] we have shown how some differentiation indices can be formulated in a very natural way within the *formal theory of differential equations* [5,6]. In particular, this leads automatically to formulations that can be applied to partial differential equations as well. In this brief note we outline how this approach can also be used to estimate the perturbation index; a detailed discussion is contained in [4]. Some further applications can be found in [8].

2. Ordinary Differential Equations

Using a geometric approach the basic theory of differentiation indices for differential algebraic equations can be presented in a simple and transparent manner (assuming a certain regularity of the problem). A differential algebraic equation can be represented by a vector field pencil $\mathcal{P}_0$ and a constraint submanifold $\mathcal{C}_0$. A solution must define a curve that both lies completely in $\mathcal{C}_0$ and is an integral curve for a vector field in $\mathcal{P}_0$. These two conditions can be satisfied simultaneously only, if $\mathcal{P}_0$ contains vector fields that are tangential to $\mathcal{C}_0$. Thus we can eliminate from the submanifold $\mathcal{C}_0$ all points where $\mathcal{P}_0$ does not contain any vector tangential to it and from $\mathcal{P}_0$ all vectors not tangential to $\mathcal{C}_0$. This leads to a smaller submanifold $\mathcal{C}_1 \subseteq \mathcal{C}_0$ and a smaller pencil $\mathcal{P}_1 \subseteq \mathcal{P}_0$.

If $\mathcal{C}_1$ is a proper submanifold of $\mathcal{C}_0$, we must check the tangency of the pencil $\mathcal{P}_1$ to it. This leads to sequences $\mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supseteq \mathcal{C}_k$ of submanifolds and $\mathcal{P}_0 \supseteq \mathcal{P}_1 \supseteq \cdots \supseteq \mathcal{P}_k$ of vector field pencils. The process stops when either an empty submanifold is reached – in this case we have an inconsistent system not possessing any solution – or when $\mathcal{C}_i = \mathcal{C}_{i+1}$. Depending on whether or not $\mathcal{P}_i = \mathcal{P}_{i+1}$ we get $k = i$ or $k = i + 1$. The final pencil $\mathcal{P}_k$ contains only vector fields tangential to the final submanifold $\mathcal{C}_k$. In most applications the final pencil consists of a single vector field, a so-called *underlying field*, otherwise we are dealing with an under-determined system.

These considerations lead to two natural concepts of differentiation indices: the *involution index* ν_i counts the total number of tangency conditions imposed in the completion process; the *determinacy index* ν_d counts the number of tangency conditions that must be imposed until the vector field pencil has shrunk to a single underlying field. ν_i is defined for any differential algebraic equation, whereas ν_d has only for not under-determined systems a finite value.

The *perturbation index* ν_p is a purely analytic concept based on an estimate for the difference of a solution of the given differential algebraic equation and a solution of a perturbed form of it [3]. The equation is of index ν_p, if the estimate depends on derivatives of the perturbation up to order $\nu_p - 1$. Campbell and Gear showed that generally there is no relation between ν_d, ν_i and ν_p [2]. This is not surprising, as ν_p is based on a perturbed form of the equation and for non-linear equations the perturbation may significantly effect the completion to involution. Thus one should study the completion of the *perturbed* form leading to the perturbed differentiation indices $\tilde{\nu}_d$ and $\tilde{\nu}_i$.

Theorem 1. *If both indices are defined for a given differential algebraic equation, then* $\tilde{\nu}_d \leq \nu_p \leq \tilde{\nu}_d + 1$.

3. Partial Differential Equations

In [7] we showed how one can easily generalise the above used index concepts from ordinary to partial differential equations. However, this was done in a rather formal manner and did not directly permit to obtain results like Theorem 1. For lack of space we cannot present here a proof of this theorem; essentially it applies Gronwall's lemma to an underlying equation. Thus the key to a generalisation of Theorem 1 to partial differential equations consists of firstly extending the notion of an underlying equation. For simplicity of the presentation, we restrict our discussion to linear first order systems; the theory holds for non-linear systems, too. If we assume that the independent variables are $\vec{x} = (x_1, \dots, x_n)$ and the dependent ones $\vec{u} = (u_1 \dots, u_m)$, we can transform the system by some elementary algebraic operations to a triangular form. As many equations as possible are solved for x_n-derivatives;[1] they form the *subsystem of class n*. Of the remaining equations we solve as many as possible for x_{n-1}-derivatives, the subsystem of class $n-1$, and so on.

We claim that the so defined subsystem of class n represents a natural generalisation of the notion of an underlying equation. This is motivated by the following observation: a vector field is underlying, if and only if the final constraint manifold $\mathcal{C}_k$ is invariant under its flow. A similar property holds for the subsystem of class n of an *involutive* system. We cannot give here a precise definition of involution; it can be found in the cited references. Besides the absence of hidden integrability conditions it requires certain combinatorial properties of the symbol.

Proposition 2. *Assume we are given a linear involutive system with analytic coefficients. Let $\vec{U}(\vec{x})$ be a smooth solution of the subsystem of class n satisfying all equations of lower class for $x_n = 0$. Then $\vec{U}(\vec{x})$ satisfies these equations for all values of x_n.*

The restriction to analytic coefficients is due to the use of Holmgren's uniqueness theorem in the proof and can be removed only under considerable technical difficulties. The assumption of a smooth solution is not necessary; the proposition could be formulated for much larger classes of solutions including weak ones. A classical example are Maxwell's equations in electrodynamics. If its evolutionary part is solved for divergence free initial data, the solution remains divergence free for all values of t. Proposition 2 shows that this is not a particular property of Maxwell's equations but holds for any involutive system.

Based on this result, Theorem 1 can now be generalised to partial differential equations. Because of the much more complicated existence theory compared with ordinary differential equations, we need additional assumptions. The simplest case arises, if we assume that the subsystem of class n can be treated by semigroup theory.

Theorem 3. *Under the same assumption as above, let furthermore the subsystem of class n be of the form $\vec{u}_{x_n} = A\vec{u}$ where A is the generator of a strongly continuous semigroup acting on an appropriately chosen Banach space. Then $\tilde{\nu}_d \leq \nu_p \leq \tilde{\nu}_d + 1$.*

Acknowledgements

This work has been supported by Landesgraduiertenförderung Baden-Württemberg and by Deutsche Forschungsgemeinschaft.

4. References

1 Brenan, K.E., Campbell, S.L., Petzold, L.R.: Numerical Solution of Initial Value Problems in Differential-Algebraic Equations; SIAM, Philadelphia 1996

2 Campbell, S.L., Gear, C.W.: The index of general non-linear DAEs; Numer. Math. **7** (1995) 173–196

3 Hairer, E., Lubich, C., Roche, M.: The Numerical Solution of Differential-Algebraic Equations by Runge-Kutta Methods; Springer, Berlin 1989

4 Hausdorf, M., Seiler, W.M.: Perturbation versus differentiation indices; Preprint Universität Mannheim 2001

5 Pommaret, J.-F.: Systems of Partial Differential Equations and Lie Pseudogroups; Gordon & Breach, London 1978

6 Seiler, W.M.: Analysis and Application of the Formal Theory of Partial Differential Equations; Ph.D. Thesis, School of Physics and Materials, Lancaster University, 1994

7 Seiler, W.M.: Indices and solvability for non-normal systems of differential equations; Computer Algebra in Scientific Computing – CASC '99, V.G. Ghanza, E.W. Mayr, E.V. Vorozhtsov (eds.), Springer, Berlin 1999, 365–385

8 Seiler, W.M.: Completion to involution and semi-discretisations; Appl. Numer. Math., to appear

Marcus Hausdorf, Werner M. Seiler, Lehrstuhl Mathematik I, Universität Mannheim, D-68131 Mannheim.
`http://www.math.uni-mannheim.de/~wms`

[1] For evolutionary systems, a natural choice for the variable x_n is the time t.