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Overdetermined Elliptic Systems

We show that the weights classically used for the definition of ellipticity are not necessary, as any differential system that is elliptic with weights becomes elliptic without weights during its completion. Furthermore, there are systems which are not elliptic for any choice of weights but whose completed form is nevertheless elliptic.

1. Introduction

An important task in the theory of partial differential equations is their classification into elliptic and hyperbolic systems. For the former ones boundary value problems are usually well-posed and their solutions show typically a very high regularity. In the latter ones a distinguished direction ("time") exists and one considers initial value problems for them; even for regular data the solution may exhibit shocks. The classification is a local problem and thus for simplicity we consider here only linear systems with constant coefficients. In more general situations, the behaviour of the system may be different at different points or along different solutions.

The classical definition of ellipticity via the principal symbol has problems when higher order equations are rewritten as first order systems. As a solution, Douglis and Nirenberg [3] introduced a weighted symbol. Cosner [2] showed later that any system elliptic with respect to some weights may be transformed into an equivalent first order system that is elliptic without weights. In our opinion, a reasonable analysis of an overdetermined system is only possible after its completion, i. e. after all its integrability conditions have been added (see also [4]). There exist different approaches to the completion (see [5] and references therein), but our results are largely independent of this choice. Full details of our results can be found in a forthcoming paper [6].

2. Square Elliptic Systems

Let $L = \sum_{0\leq|\mu|\leq q} A_\mu \partial^\mu$ be a linear differential operator of order q with coefficients $A_\mu \in \mathbb{R}^{m\times m}$. Here $\mu \in \mathbb{N}_0^n$ is a multi index with one entry for each independent variable. The *principal symbol* is $\sigma L = \sum_{|\mu|=q} A_\mu \xi^\mu$ for a real vector $\xi \in \mathbb{R}^n$, i. e. the highest order part considered as a polynomial in ξ. The operator L is *elliptic*, if $\det \sigma L \neq 0$ for all $\xi \neq 0$. For the *weighted* principal symbol one chooses integer weights s_i for each equation and t_j for each unknown function. If q_{ij} is the maximal order of a derivative of the jth unknown function in the ith equation, then they must satisfy $s_i + t_j \geq q_{ij}$. The weighted principal symbol is now again a matrix with polynomial entries: $(\sigma_w L)_{ij} = \sum_{|\mu|=s_i+t_j} A_\mu \xi^\mu$. The operator L is *DN-elliptic*, if weights exist such that $\det \sigma_w L \neq 0$ for all $\xi \neq 0$.

The prototypical example of an elliptic system is Laplace's equation $Lu = u_{xx} + u_{yy} = 0$. Its principal symbol is $\sigma L = \xi_x^2 + \xi_y^2$ and only vanishes for $\xi = 0$. Rewriting it as a first order system, we obtain $v = u_x$, $w = u_y$ and $v_x + w_y = 0$. With the weights $s_1 = s_2 = -1$, $s_3 = 0$, $t_1 = 2$, $t_2 = t_3 = 1$, we find for the corresponding operator L_1

$$\sigma L_1 = \begin{pmatrix} \xi_x & 0 & 0 \\ \xi_y & 0 & 0 \\ 0 & \xi_x & \xi_y \end{pmatrix}, \qquad \sigma_w L_1 = \begin{pmatrix} \xi_x & -1 & 0 \\ \xi_y & 0 & -1 \\ 0 & \xi_x & \xi_y \end{pmatrix}. \tag{1}$$

As $\det \sigma L_1 \equiv 0$, we obtain the paradoxical result that seemingly the first order form of Laplace's equation is not elliptic, although the reduction to first order does not change the properties of the solution space. In contrast, we obtain for the used weights that $\det \sigma_w L_1 = \xi_x^2 + \xi_y^2 = \sigma L$. Hence L_1 is DN-elliptic.

3. Overdetermined Elliptic Systems

We proceed to the general case with $A_\mu \in \mathbb{R}^{k\times m}$ where we always assume that $k \geq m$, i. e. that we have at least as many equations as unknown functions. The definition of the (weighted) principal symbol is not affected by this generalisation. The operator L is *(DN-)elliptic*, if its (weighted) principal symbol is injective for all non-vanishing vectors ξ. In the case $k = m$ this is equivalent to the above condition on the determinant.

If we consider again Laplace's equation in its first order form, then we see that there exists a hidden integrability condition: $v_y = w_x$. Adding this equation yields a complete system whose principal symbol is injective for any non-vanishing vector ξ. Thus there is no need to introduce weights for checking the ellipticity. This is not an accident but a general fact, as the following theorem asserts.

T h e o r e m 1. *Let the operator L be DN-elliptic. Then its completion L' is elliptic.*

The proof of this result proceeds in two steps. One first shows that if the operator L is DN-elliptic, then it may be transformed via differentiations and algebraic manipulations into an equivalent elliptic operator $\hat{L}$. In a second step one shows that ellipticity is preserved during the completion and that all rows in $\hat{L}$ are direct differential consequences of rows in the completed operator L'. Hence L' is elliptic, too. In the second step it does not really matter which approach to the completion is taken, as any reasonable approach leads to a completed operator L' with the required properties.

As a second example we consider the system $Lu = \nabla \times u + u = 0$. One readily checks that it is not possible to find any weights such that this system is DN-elliptic. However, if we add the obvious integrability condition $\nabla \cdot u = 0$, we obtain a system whose principal symbol is injective for any non-vanishing vector ξ. Thus the system is elliptic, but the approach via weights is not able to detect this.

4. Conclusions

Weights partially simulate the effect of a completion. They allow lower order terms to enter the principal symbol and in some cases these are precisely the terms leading to integrability conditions. This happens for Laplace's equation in first order form. However, in general not all relevant terms can be reached via weights, as our second example demonstrates. In any case we may conclude that weights are neither necessary nor sufficient for deciding ellipticity.

Completion to involution is a basic operation with overdetermined systems that is not only important for the classification but for any subsequent analysis like existence and uniqueness theory or the numerical treatment. While Cosner's results [2] are based on an ad hoc construction; our results rely on a standard computation. They show that the sole problem in detecting ellipticity lies in taking all integrability conditions into account.

Adding all appearing integrability conditions one can show that the reduction to first order preserves ellipticity. The same holds for the Drach transformation to a system in only one unknown function (provided one adds a suitable "gauge fixing"). This allows us to apply Agmon's regularity results [1] entailing in particular that any elliptic system with smooth coefficients is hypoelliptic.

Acknowledgements

This work has been partially supported by Deutsche Forschungsgemeinschaft.

5. References

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