Indices and Solvability for General Systems of Differential Equations

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Abstract. We consider general systems of ordinary and partial differential equations from a geometric point of view. This leads to simple interpretations of various index concepts introduced for differential algebraic equations. Especially, we obtain natural generalisations of these concepts to partial differential equations.

1 Introduction

Most of the literature on differential equations is concerned with so-called *normal* systems. In the case of ordinary differential equations this means that the system can be written in explicit form $\dot{u} = \phi(t, u)$ for some vector-valued function $u(t)$. For partial differential equations the condition of normality is more restrictive: it requires the existence of a distinguished independent variable t such that the system can be written in the form $u_t = \phi(t, x, u, u_x)$ where x represents the remaining independent variables (i. e. they satisfy the conditions of the Cauchy-Kowalevsky theorem).

From a theoretical point of view the restriction to normal systems is not very severe in the case of ordinary differential equations. Neglecting singularities any system can at least locally be transformed into an explicit form. In concrete applications one often either cannot explicitly construct or for some reasons does not want to perform this transformation. As a consequence, *differential algebraic equations* [1,8] or more generally *implicit equations* of the form $\Phi(t, u, \dot{u}) = 0$ have found much attention lately.

Simplifying a bit, a system of ordinary differential equations is not normal, if it contains equations of different order. In the usually considered case of first order systems this means that some of the equations are algebraic. The transformation to a normal system is performed by solving these *constraints* and thus eliminating some of the dependent variables. In contrast, a non-normal system of partial differential equations contains usually equations of different class (see Sect. 2.2) and not of different order. In general, no transformation to a normal system exists.

Non-normal systems of partial differential equations are either under- or overdetermined and occur in many applications. For example the field equations of a gauge theory cannot be normal, as the presence of the gauge symmetry implies underdeterminacy. Inverse problems like the construction of a potential for a given force often lead to overdetermined systems. A characteristic property of such systems is the possible existence of integrability conditions. The process of constructing all non-trivial integrability conditions for a given system is often called *completion*.

Already in the late 19^{th} century mathematicians started to study general systems of differential equations and to design completion algorithms. Most notable among these classical approaches is the Janet-Riquier theory [9]; the first geometric approach was the Cartan-Kähler theory [2]. We will use the *formal theory* [18,19] combining elements of both approaches.

All these theories have remained comparatively "exotic". Restricted to ordinary differential equations they become straightforward. Thus it is not surprising that for this special case they have been rediscovered many times in various fields. For differential algebraic equations there exist at least three more or less independently developed approaches [21,22,33] which may be considered as specialisations of the general completion theory. For constrained Hamiltonian systems the same holds for the Dirac theory [6].

A unifying treatment of these two fields was given in [30]. The purpose of this article is to extend the results presented there to infinite dimensional systems. We will give geometric versions of several index definitions proposed in the literature which can be directly applied to partial differential equations.

In order to make this article as self-contained as possible, the next section provides an introduction into formal theory. Section 3 contains various index definitions for general systems of differential equations plus a discussion of their meaning. Relations between the different indices are studied in Sect. 4. Finally, some conclusions are given.

2 Formal Theory of Differential Equations

It is beyond the scope of this article to provide an in-depth introduction to formal theory. For this we must refer to the literature [18,19]. We will give here a brief overview following the presentation in [25]. Section 2.4 contains some simple examples; Sect. 2.5 specialises the theory to ordinary differential equations where it becomes much simpler.

2.1 Jet Bundle Formalism

Although the whole theory can be presented in a coordinate-free manner, we will use throughout local coordinates. Our notations are as follows. The independent variables of the differential equation are denoted by $x^1, \dots, x^n$; they are local coordinates for the base space X. The dependent variables or unknown functions are denoted by $u^1, \dots, u^m$. Together with the x^i they form bundle coordinates for a fibre bundle $\pi : \mathcal{E} \to X$. For derivatives we use a multi-index notation. A multi-index μ is an ordered tuple $[\mu_1, \dots, \mu_n]$ of integers; its length is defined as $|\mu| = \sum \mu_i$. Then we denote the derivative $\partial^{|\mu|} u^\alpha / \partial(x^1)^{\mu_1} \cdots \partial(x^n)^{\mu_n}$ by p^α_μ.

The q-th order *jet bundle* $\mathcal{J}_q\mathcal{E}$ is again a fibre bundle with bundle coordinates $(x^i, u^\alpha, p^\alpha_\mu)$ with $|\mu| \leq q$. A point in such a space may be considered as an equivalence class of functions where two functions are equivalent, if they possess identical Taylor series up to order q. A *differential equation* $\mathcal{R}_q$ of order q is defined as a (fibred) submanifold of $\mathcal{J}_q\mathcal{E}$. Locally, we may describe it by some equations $\Phi^\tau(x^i, u^\alpha, p^\alpha_\mu) = 0$ where $\tau = 1, \ldots, p$ and recover thus the usual point of view. Note that we do not require that $p = m$! Furthermore this definition makes no distinction between a single equation and a system.

Jet bundles of different order are naturally related by projections: If $r > q$, we define the map $\pi^r_q : \mathcal{J}_r\mathcal{E} \to \mathcal{J}_q\mathcal{E}$ which simply "forgets" all derivatives of higher order than q. For a given differential equation $\mathcal{R}_q \subset \mathcal{J}_q\mathcal{E}$ there exist two natural operations: *prolongation* yields an equation $\mathcal{R}_{q+1} \subset \mathcal{J}_{q+1}\mathcal{E}$ of order $q+1$; *projection* leads to an equation $\mathcal{R}^{(1)}_{q-1} = \pi^q_{q-1}(\mathcal{R}_q) \subset \mathcal{J}_{q-1}\mathcal{E}$ of order $q-1$. More generally, we will denote by $\mathcal{R}^{(s)}_q$ the equation obtained by s prolongations with subsequent projection: $\mathcal{R}^{(s)}_q = \pi^{q+s}_q(\mathcal{R}_{q+s})$.

In the classical language, prolongation corresponds to differentiating each equation $\Phi^\tau = 0$ with respect to all independent variables. To obtain a local representation of a projected system is in general rather difficult, especially for nonlinear equations, as it requires algebraic elimination.

2.2 Involutive Differential Equations

A key concept in formal theory is *involution*. We approach it in a computational manner considering the order by order construction of the general power series solution for a given differential equation $\mathcal{R}_q$. We make the ansatz $u^\alpha(x) = \sum_{|\mu|\geq 0} \frac{a^\alpha_\mu}{\mu!}(x-x_0)^\mu$, enter it into a local description $\Phi^\tau = 0$ and evaluate the arising equations at $x = x_0$. This yields an algebraic system for the Taylor coefficients a^α_μ up to order q. Then we do the same with the prolonged equation $\mathcal{R}_{q+1}$ and get a linear system (usually underdetermined) for the Taylor coefficients of order $q+1$. This procedure can be repeated with higher prolongations $\mathcal{R}_{q+r}$ as often as one wishes.

This approach produces a valid series solution only, if at no order integrability conditions appear. Otherwise we cannot be sure whether we have correctly identified all conditions the Taylor coefficients must satisfy. Assume for example that we stop at some order $q_0 > q$ but that at a higher order $q_1 > q_0$ an integrability condition of, say, order $q+1$ was hidden. This additional condition on the coefficients of order $q+1$ was ignored in our construction and hence we obtained a series with too many free coefficients.

The differential equation $\mathcal{R}_q$ is called *formally integrable*, if $\mathcal{R}^{(r)}_q = \mathcal{R}_q$ holds for all $r \geq 0$. In other words, no matter how often we prolong such an equation, we never find an integrability condition and thus the above described order by order construction of power series solutions can be pursued without problems. Unfortunately, no finite criterion is known to decide whether or not a given differential equation $\mathcal{R}_q$ is formally integrable. This leads to the idea of involution: an involutive equation is formally integrable but possesses in addition further properties which allow for a finite test.

The easiest way to understand involution is to pose the following question: if we use the above outlined construction of the general power series solution, how many Taylor coefficients at some order $q+r$ will be determined by the differential equation and how many can be chosen freely. The former ones are called *principal coefficients*; the latter ones *parametric coefficients*.

The answer is encoded in the *symbol* $\mathcal{M}_q$ of the differential equation $\mathcal{R}_q$. The symbol is a matrix, namely the principal part of the linearisation of $\mathcal{R}_q$. Its formal definition goes as follows. Let a local description of $\mathcal{R}_q$ be given by $\Phi^\tau(x,u,p)=0$. For each derivative p^α_μ of order q we introduce an unknown a^α_μ (note that we consider only the derivatives of maximal order). The symbol is the matrix of the following linear system in these unknowns a^α_μ

$$\mathcal{M}_q\ :\ \left\{\ \sum_{\alpha=1}^{m}\sum_{|\mu|=q}\frac{\partial\Phi^\tau}{\partial p^\alpha_\mu}a^\alpha_\mu=0\,,\qquad \tau=1,\dots,p\,.\right. \tag{1}$$

The symbol $\mathcal{M}_{q+r}$ of the prolonged system $\mathcal{R}_{q+r}$ appears in our construction of power series solutions as the matrix of the linear system for the Taylor coefficients a^α_μ of order $q+r$. Thus the number of principal coefficients of order $q+r$ is given by $\operatorname{rank}\mathcal{M}_{q+r}$. In order to predict this number based solely on the symbol $\mathcal{M}_q$, we introduce the *class* of a multi-index $\mu=[\mu_1,\dots,\mu_n]$ as the smallest index k such that $\mu_k\neq 0$.

We order the columns of $\mathcal{M}_q$ – corresponding to the unknowns a^α_μ – so that columns for unknowns with a multi-index of higher class are always to the left of columns for unknowns with a multi-index of lower class. Otherwise the ordering does not matter. Then we compute a row echelon form of $\mathcal{M}_q$ and denote by $\beta^{(k)}_q$ the number of rows where the pivot sits in a column for an unknown with a multi-index of class k.

The symbol $\mathcal{M}_q$ is called *involutive*, if $\operatorname{rank}\mathcal{M}_{q+1}=\sum_{k=1}^n k\beta^{(k)}_q$. This definition sounds rather mysterious, but one can rather easily see that this sum is a lower bound for $\operatorname{rank}\mathcal{M}_{q+1}$ (this follows from an analysis of the pivots of $\mathcal{M}_{q+1}$). It is also easy to show that the prolongation of an involutive symbol is again involutive. Thus for involutive symbols $\mathcal{M}_q$ one can derive by induction a closed form expression for $\operatorname{rank}\mathcal{M}_{q+r}$ depending only on the $\beta^{(k)}_q$.

The differential equation $\mathcal{R}_q$ is *involutive*, if it is formally integrable and if its symbol is involutive. One can show that if $\mathcal{M}_q$ is involutive and $\mathcal{R}^{(1)}_q=\mathcal{R}_q$, i.e. no integrability conditions appear in the next prolongation, then $\mathcal{R}_q$ is formally integrable and thus involutive. This yields an effectively checkable criterion for involution: involution of the symbol is a matter of some linear algebra and we need only one prolongation. The numbers $\beta^{(k)}_q$ encode the size of the solution space [26].

2.3 Solvability and Completion

The most important property of an involutive system $\mathcal{R}_q$ is that it possesses formal power series solutions. In the case of analytic systems we can even prove the convergence of the series and obtain an existence and uniqueness theorem for analytic solutions: the *Cartan-Kähler theorem*. Its proof sheds some further light on the above analysis of symbols based on the class.

We assume for simplicity that we have a first order equation. Then equations of lower order are algebraic constraints and can again (at least in principle) be treated by solving them and eliminating some dependent variables. The remaining equations are collected into subsystems according to their class. If we consider in the equations of class k the independent variables $x^{k+1}, \dots, x^n$ as parameters (which is possible as by definition of the class no derivatives with respect to these variables appear), they form a normal system and the Cauchy-Kowalevsky theorem yields the existence and uniqueness of analytic solutions. These solutions are now taken as initial data for the normal system formed by the equations of class $k+1$ and so on.

This construction could actually be performed for any formally integrable system. However, there remains one final point. After eventually treating the subsystem of class n we obtain solutions depending on all variables x^i. But in the previous steps we always considered some of them as constant parameters and it is not clear whether the final solutions still solve all equations of lower class. But if the system is involutive, one can prove by one more application of the Cauchy-Kowalevsky theorem that this is indeed the case.

The question naturally arises what happens, if one encounters a system which is not involutive. The (perhaps surprising) answer is given by the *Cartan-Kuranishi theorem*. It states that (under some regularity assumptions) any differential equation can either be made involutive by a finite number of prolongations and projections or it is inconsistent and has thus no solutions. This process is called *completion*.

The proof of the Cartan-Kuranishi theorem provides a simple completion algorithm consisting of two nested loops. The inner one prolongs the system until its symbol becomes involutive. The outer one then checks whether integrability conditions have occurred. If yes, they are added and the whole procedure starts again. Otherwise we have reached an involutive system. It is relatively straightforward to implement this algorithm in a computer algebra system.[1] An Axiom implementation is described in [24,27].

The Cartan-Kuranishi theorem represents a rather deep result in formal theory. The main point is to show that the above outlined algorithm indeed terminates. Especially for the inner loop this is highly nontrivial. The fact that any symbol becomes involutive after a finite number of prolongations is sometimes called the *prolongation theorem*. Its proof is usually based on cohomological techniques. The termination of the outer loop stems from a simple Noetherian argument.

[1] There are a number of technical subtleties which pose some real problems.

In any case we can conclude that the question of (local) solvability for analytic equations is completely answered by our completion algorithm. If no inconsistencies, i.e. integrability conditions of the form $0 = 1$, appear, the equation possesses analytic solutions. Note that analyticity is a necessary condition here! Already for smooth equations further obstacles to local solvability may appear (of so-called Lewy type [11, Chapt. 8]).

Completion (by our algorithm or any other like the ones coming from Janet-Riquier theory) represents essentially the only *systematic* way to prove the consistency of a general system of differential equations. For many systems consistency can be simply shown by exhibiting trivial solutions like $u^\alpha \equiv 0$, but obviously this cannot be done systematically.

2.4 Examples

As a first simple example, we consider the first order differential equation $\mathcal{R}_1$ defined by the two equations $u_z - yu_x = 0$ and $u_y = 0$. If we take the natural ordering $x^1 = x$, $x^2 = y$ and $x^3 = z$, the symbol $\mathcal{M}_1$ is the matrix $\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \end{pmatrix}$ which is already in row echelon form. Its first row is of class 3 and the second one of class 2; hence $\beta_1^{(3)} = \beta_1^{(2)} = 1$ and $\beta_1^{(1)} = 0$. Prolongation leads to the second order system $\mathcal{R}_2$ defined by the six additional equations $u_{zz} - yu_{xz} = u_{yz} - yu_{xy} - u_x = u_{xz} - yu_{xx} = 0$ and $u_{xy} = u_{yy} = u_{yz} = 0$. It is easy to see that all these equations are independent and that one can eliminate all second order derivatives in one of the equations. Thus $\operatorname{rank} \mathcal{M}_2 = 5 = 3\beta_1^{(3)} + 2\beta_1^{(2)} + \beta_1^{(1)}$ and consequently $\mathcal{M}_1$ is involutive and we have already finished the inner loop of the completion algorithm.

However, the elimination yields the integrability condition $u_x = 0$. Thus $\mathcal{R}_1$ is not involutive and we must start again with the projected system $\mathcal{R}_1^{(1)}$ defined by the three equations $u_x = u_y = u_z = 0$. It is left as an exercise to the reader to show that $\mathcal{R}_1^{(1)}$ is involutive. Here we have an example of a so-called finite type system with a finite-dimensional solution space, as its general solution is obviously $u(x, y, z) \equiv \text{Const.}$

A classical example due to Janet where several iterations through both loops of the completion algorithm are needed is given by the second order system $\mathcal{R}_2$ defined by $u_{zz} - yu_{xx} = 0$ and $u_{yy} = 0$. Full details to this example can be found in [20]; here we can only sketch the main steps. The symbol becomes involutive after two prolongations. However, there arises the integrability condition $u_{xxy} = 0$. The symbol of the corresponding system $\mathcal{R}_3^{(1)}$ becomes involutive again only after two prolongations which also lead to an integrability condition: $u_{xxxx} = 0$. The corresponding system $\mathcal{R}_4^{(2)}$ is formally integrable but not yet involutive, as its symbol needs one more prolongation in order to become involutive. Thus our completion algorithm ends with the system $\mathcal{R}_5^{(2)}$ from which it is easy to deduce that it is also of finite type with a 12-dimensional solution space.

A well-known example of a non-normal system are *Maxwell's equations* of electrodynamics. In field strength formulation they represent a system of eight differential equations for six field components:

$$\epsilon\partial_t \boldsymbol{E} - \operatorname{curl} \boldsymbol{H} = 0\,, \quad \mu\partial_t \boldsymbol{H} + \operatorname{curl} \boldsymbol{E} = 0\,, \tag{2}$$

$$\operatorname{div} \boldsymbol{E} = 0\,, \quad \operatorname{div} \boldsymbol{H} = 0\,. \tag{3}$$

If we take the time t as independent variable x^4, we have six equations of class 4, namely the six evolution equations (2). The two divergence laws (3) are of class 3. There is no need for a completion of Maxwell's equations, as one can easily check that they form already an involutive system.

In the numerical solution of the initial value problem for Maxwell's equations one often follows precisely the strategy of the proof of the Cauchy-Kowalevsky theorem outlined above. One first determines a time-independent solution of (3) and uses it as initial data for the hyperbolic system (2). Involution of Maxwell's equations guarantees that the such obtained solutions satisfy the divergence laws at all times t. Of course, this holds only for the exact solutions! Numerical solutions often violate (3) and the sometimes observed problem of spurious solutions also seems to be related to neglecting the divergence laws [10].

As a final example for some of the effects appearing in general systems of partial differential equations we consider the system

$$\partial_{xx} u - \partial_{xt} v = 0\,, \quad \partial_{tt} v - \partial_{xt} u = 0\,. \tag{4}$$

With two equations for two unknown functions, (4) would traditionally be considered as a welldetermined system. In fact, it is underdetermined: if u, v are any solution of (4), $u + \partial_t \Lambda$, $v + \partial_x \Lambda$ are also solutions for *any* function $\Lambda(t, x)$ and we can choose either u or v completely arbitrary.

The simple explanation is that while (4) is involutive, it is not normal! No matter how we order the independent variables, we have always one equation of class 1. In fact, (4) is nothing else than Maxwell's equations in $1 + 1$ dimensions in potential instead of field strength formulation. And it is well-known that in this form we have a gauge symmetry. But the field equations of a gauge theory can never be normal and are always underdetermined.

2.5 Ordinary Differential Equations

Formal theory becomes rather simple when applied to ordinary differential equations. The most important simplification comes from the fact that the symbol of an ordinary differential equations is always involutive. This is rather trivial, as there is only one independent variable and all equations are of the same class. Hence in our completion algorithm we can skip the inner loop.

The completion takes the following simple form. All equations are prolonged. Then one checks for integrability conditions. If some arise, they are added. Now we must check whether the prolongation of these new equations leads to further integrability conditions. If not the completion has finished. In the case of a semi-explicit system the completion can be further simplified. Here it suffices to prolong only the constraint equations, as the prolongations of the true differential equations drop out during the subsequent projection.

This can be seen as follows. A semi-explicit system has the form

$$\mathcal{R}_1 \,:\, \begin{cases} \dot{u} = \phi(t,u,v)\,, \\ 0 = \psi(t,u,v) \end{cases} \tag{5}$$

where u, v may be vectors. The prolonged system is given by

$$\mathcal{R}_2 \,:\, \begin{cases} \dot{u} = \phi(t,u,v)\,, & 0 = \psi(t,u,v)\,, \\ \ddot{u} = \phi_t + \phi_u \dot{u} + \phi_v \dot{v}\,, & 0 = \psi_t + \psi_u \dot{u} + \psi_v \dot{v}\,, \end{cases} \tag{6}$$

Projecting back to the $\mathcal{J}_1\mathcal{E}$ eliminates the equation for $\ddot{u}$. Thus

$$\mathcal{R}_1^{(1)} \,:\, \begin{cases} \dot{u} = \phi(t,u,v)\,, \\ 0 = \psi_t + \psi_u \phi + \psi_v \dot{v}\,, \\ 0 = \psi(t,u,v)\,. \end{cases} \tag{7}$$

One obtains the same system by only prolonging the constraints $\psi = 0$. The equations in (7) are not necessarily algebraically independent, but it requires only linear algebra to transform $\mathcal{R}_1^{(1)}$ into semi-explicit form. Thus in the next step of the completion it suffices again to prolong only the constraints. Geometrically, the constraints define the submanifold $\mathcal{R}_0^{(1)} = \pi_0^1(\mathcal{R}_1) \subset \mathcal{E}$.

In this simple form the completion algorithm has been rediscovered many times. A prominent example is given by the Dirac algorithm for constrained Hamiltonian systems [6]. Although the prolongations are "disguised" behind Poisson brackets, the Dirac algorithm corresponds for finite-dimensional systems exactly to completing the Hamiltonian equations of motion to an involutive system [31]. But for field theories the Dirac algorithm does in general not suffice to prove consistency of the field equations, as it does not check all possible integrability conditions.

For general differential algebraic equations similar algorithms have already been presented several times [21,22,33]. Their results are more or less identical with ours for systems where all approaches can be applied. Some of them go beyond the results presented here in admitting singularities or less regular solutions. We have excluded many kinds of singularities by defining a differential equation as a submanifold.

Most of these articles propose a slight variant of the completion algorithm. We prolong first and then project, whereas they reverse the operations and first project and then prolong. But this yields equivalent results. We showed already above that in the case of a semi-explicit system, it suffices to prolong only the constraints. Their projection can be interpreted as a kind of transformation into a semi-explicit form, as it produces just these constraints, i.e. the system $\mathcal{R}_0^{(1)}$. Hence if they subsequently prolong these, they obtain exactly the same result as if they first prolonged and then projected.

3 Index Definitions

In the literature on differential algebraic equations there is an abundance of index definitions. We do not attempt to give an overview; a general discussion of indices was recently given by Campbell and Gear [3]. We rather concentrate on the two mainly used types: differentiation and perturbation indices.

Given an implicit first order ordinary differential equation $\mathcal{R}_1$ locally described by $\Phi^\tau(t, u^\alpha, \dot{u}^\alpha) = 0$, we can roughly explain the ideas behind these two classes of indices as follows. *Differentiation indices* count the number of prolongations needed to obtain an equation with some specific property. For "the" differentiation index this property is that all first order derivatives $\dot{u}^\alpha$ can be computed explicitly as a function of the basic variables t, u^α.

Perturbation indices stem from estimates for the difference between solutions of $\mathcal{R}_1$ and solutions of a perturbed equation $\tilde{\mathcal{R}}_1$. The index says up to which order derivatives of the perturbations enter the estimate. Typically, differentiation indices are easier to compute than perturbation indices, but the latter ones are more important for the numerical analysis. Thus there is much interest in relating the two classes of indices.

3.1 Differentiation and Determinacy Index

The most widely used index in the theory of differential algebraic equations is "the" differentiation index. It is often defined via the so-called derivative array of $\mathcal{R}_1$. The k^{th} array consists of the equations up to the k-th prolongation, i.e. it forms a local representation of $\mathcal{R}_{1+k}$.

Classically, one defines the differentiation index of a first order system as the smallest number ν such that all first order derivatives can be uniquely determined as functions of the independent and the dependent variables from the equations in the ν^{th} derivative array. In the language of formal theory this idea can be expressed intrinsically using the symbol.

Definition 1. The *differentiation index* of the differential equation $\mathcal{R}_q$ is the smallest integer $\nu = r + s$ such that $\dim \mathcal{M}^{(s)}_{q+r} = 0$.

In this form the definition can be applied to partial differential equations, too. Applied to first order differential algebraic equations it coincides with the usual one. Then $q = 1$ and, as discussed in the Sect. 2.5, we can always choose $r = 0$. The vanishing of the symbol $\mathcal{M}^{(\nu)}_1$ means nothing else than that all first order derivatives are principal and thus determined by equations.

However, for partial differential equations this definition is not very useful, since for most of them one never obtains a vanishing symbol. A vanishing symbol signals a finite dimensional solution space and this occurs only for very overdetermined equations. Consequently, we rephrase the definition in such a way that it remains unchanged for ordinary differential equations but makes more sense for partial differential equations.

Definition 2. The *determinacy index* of the differential equation $\mathcal{R}_q$ in n independent and m dependent variables is the smallest integer $\nu_{\mathrm{d}} = r + s$ such that for $\mathcal{R}^{(s)}_{q+r}$ we find $\beta^{(n)}_{q+r} = m$.

For ordinary differential equations Def. 2 is equivalent to Def. 1, since for them $n = 1$ and the dimension of the symbol is $m - \beta^{(1)}_1$. For partial differential equations the condition in Def. 2 is much weaker than the one in Def. 1. Only underdetermined equations like (4) do not have a finite determinacy index. This fact explains the new name: the determinacy index ν_{d} counts the number of prolongations needed to decide whether $\mathcal{R}_q$ is underdetermined.[2] In general, it contains no further information.

Especially, ν_{d} contains generally *no* information about the solvability of the equation. This can easily be seen by considering the following differential algebraic equation in m dependent variables u^α

$$\mathcal{R}_1 : \begin{cases} \dot{u}^\alpha = \phi^\alpha(t, u^\beta)\,, & \alpha = 1, \dots, m\,, \\ \psi^\tau(t, u^\beta) = 0\,, & \tau = 1, \dots, m' < m\,. \end{cases} \tag{8}$$

Obviously for such an equation $\nu = \nu_{\mathrm{d}} = 0$. But prolonging the algebraic constraints may lead to further constraints or even to an inconsistency.

The determinacy index ν_{d} is determined during the completion algorithm described in Sect. 2.3. In general, one does not have to go through the full algorithm, if one is only interested in ν_{d}, since the condition $\beta^{(n)}_{q+r} = m$ is typically satisfied already at an earlier stage.

For ordinary differential equations there exists a simple geometric interpretation of the determinacy index ν_{d}. As soon as one has prolonged the equation ν_{d} times, one can compute a vector field such that any solution of the differential algebraic equation corresponds to an integral curve of it (but not necessarily vice versa!). Reich [22] called this uniquely determined field the *corresponding vector field*.

For its geometric construction we consider the equation $\mathcal{R}^{(\nu_{\mathrm{d}})}_1$ obtained after ν_{d} prolongations (and subsequent projections). Its algebraic part is described intrinsically by $\mathcal{R}^{(\nu_{\mathrm{d}}+1)}_0 = \pi^1_0(\mathcal{R}^{(\nu_{\mathrm{d}})}_1) \subset \mathcal{E}$. We restrict the inverse of the projection $(\pi^1_0)^{-1}$ to $\mathcal{R}^{(\nu_{\mathrm{d}}+1)}_0$. Its image at each point is m-dimensional, as the fibre dimension of $\mathcal{J}_1\mathcal{E}$ over $\mathcal{E}$ is m for ordinary differential equations. The dimension of the intersection of this image with $\mathcal{R}^{(\nu_{\mathrm{d}})}_1$ is $\dim \mathcal{M}^{(\nu_{\mathrm{d}})}_1$. Since ν_{d} was defined such that this symbol vanishes, the intersection consists of a single point (under some mild regularity assumptions) and can naturally be identified with a (non-autonomous) vector field.

For partial differential equations there does not exist an analogue to the corresponding vector field. Here we have only the concept of *local solvability*. Recall that a point in the q^{th} order jet bundle $\mathcal{J}_q\mathcal{E}$ may be interpreted as

[2] An involutive differential equation $\mathcal{R}_q$ is underdetermined, iff $\beta^{(n)}_q < m$ [25,28].

a power series truncated at order q. For an involutive equation $\mathcal{R}_q$ to every point $P \in \mathcal{R}_q$ there exists at least one formal power series solution whose truncation at order q corresponds to P. The striking difference to ordinary differential equations is that there this series is unique whereas for partial differential equations there exist in general infinitely many such series.

The corresponding vector field should not be confused with a so-called *underlying equation* which may also be considered as a vector field in the case of ordinary differential equations. While we have seen that the corresponding vector field is unique but only defined on the constraint manifold $\mathcal{R}_0^{(\nu_\mathrm{d}+1)}$, an underlying equation is never unique but defined everywhere.

For an arbitrary differential equation $\mathcal{R}_q$ in n independent and m dependent variables, we call another equation $\tilde{\mathcal{R}}_q$ underlying, if it satisfies: (i) $\mathcal{R}_q \subset \tilde{\mathcal{R}}_q$, (ii) $\tilde{\mathcal{R}}_{q-1}^{(1)} = \mathcal{J}_{q-1}\mathcal{E}$, (iii) $\beta_q^{(n)} = m$ and all other $\beta_q^{(k)}$ vanish. The first condition implies that every solution of $\mathcal{R}_q$ is also a solution of $\tilde{\mathcal{R}}_q$ (but not necessarily vice versa); the second condition says in a geometric way that there are no lower order equations in $\tilde{\mathcal{R}}_q$ and the third condition requires that $\tilde{\mathcal{R}}_q$ is normal. (iii) can be omitted in the case of ordinary differential equations, as there it is implied by (ii). This is not the case for partial differential equations. In the case of Maxwell's equations the evolution part (2) may be considered as an underlying equation.

3.2 Integrability and Involution Index

The next index concept we consider is directly related to the solvability of the considered differential equation. As already mentioned several times a formally integrable equation possesses at least formal power series solutions. In the analytic case the Cartan-Kähler theorem even yields the convergence of these series. In view of these facts it appears more naturally to consider the following indices instead of ν_d.

Definition 3. The *formal integrability index* of the differential equation $\mathcal{R}_q$ is the smallest integer $\nu_\mathrm{f} = r + s$ such that $\mathcal{R}_{q+r}^{(s)}$ is formally integrable. The *involution index* of the differential equation $\mathcal{R}_q$ is the smallest integer $\nu_\mathrm{i} = r + s$ such that $\mathcal{R}_{q+r}^{(s)}$ is involutive.

In principle, ν_f prolongations suffice to prove the consistency of a given differential equation. However, in practice we need $\nu_\mathrm{i} \geq \nu_\mathrm{f}$ prolongations, since we know no finite criterion to decide formal integrability except completion to an involutive equation. In the case of ordinary differential equations $\nu_\mathrm{f} = \nu_\mathrm{i}$, as their symbols are always involutive (cf. Sect. 2.5).

We outlined above for differential algebraic equations the construction of the corresponding vector field. It can be performed as soon as the symbol vanishes. If $\nu_\mathrm{f} > \nu_\mathrm{d}$ and we apply this construction to the different equations $\mathcal{R}_1^{(\nu_\mathrm{d})} \supset \mathcal{R}_1^{(\nu_\mathrm{d}+1)} \supset \cdots \supset \mathcal{R}_1^{(\nu_\mathrm{f})}$, this field does not really change; only its domain of definition shrinks at each step as more and more constraints are added. Once $\mathcal{R}_1^{(\nu_\mathrm{f})}$ is reached (and it is not empty), every integral curve of the corresponding vector field represents a solution of the differential algebraic equation. At the previous steps only the converse holds. Thus the field defined by $\mathcal{R}_1^{(\nu_\mathrm{f})}$ represents a geometric version of what is often called *state space form* of the differential algebraic equation.

For linear differential algebraic equations the integrability index ν_f coincides with the *strangeness index* introduced by Kunkel and Mehrmann [12,13] (they called the co-dimension of the constraint manifold $\mathcal{R}_1^{(0)}$ the strangeness of $\mathcal{R}_1$). To some extent ν_f may also be considered as a geometric version of the *uniform differentiation index* ν_ud introduced by Campbell et al. [3,4]. When ν_ud is defined, it ensures like ν_f the consistency of the equations and thus the existence and uniqueness of solutions.

There are two main differences between ν_f and ν_ud. The fact that the definition of ν_ud is based on $\mathcal{R}_{1+s}$ instead of $\mathcal{R}_1^{(s)}$ (i. e. that no projections are performed) is not important, as it does not influence the obtained index values. The first difference is that Condition (A4) of [3,4] excludes the possibility of trivial integrability conditions, as they always appear in overdetermined systems. Thus ν_ud is not defined for systems of the form (8).

The second and more important difference is that in formal theory ranks are always evaluated *on* the submanifolds $\mathcal{R}_1^{(s)}$, whereas the definition of ν_ud requires constant ranks in open neighbourhoods of solutions. Effectively this forbids algebraic manipulations of the equations and can lead to very different values of the two indices.

We demonstrate this effect on a simple three-dimensional system taken from [3]: $\sin(\dot{y})y + x = 0$, $\sin(\dot{z})z + y = 0$, and $z = 0$. The submanifold $\mathcal{R}_1$ defined by these equations can equivalently be described by the equations $x = y = z = 0$. Thus it is trivial to see that $\mathcal{R}_1$ becomes formally integrable after one prolongation and $\nu_\mathrm{f} = 1$. As the two sets of equations are of course not equivalent in an open neighbourhood of $\mathcal{R}_1$, one obtains a different value for the uniform differentiation index, namely $\nu_\mathrm{ud} = 3$.

3.3 Perturbation Index

The perturbation index was introduced by Hairer et al. [7]; it is an analytic concept and thus cannot be geometrically expressed like the above differentiation indices. But we will show in Sect. 4.2, how formal theory can be useful for estimating the perturbation index.

Hairer et al. introduced this index only for differential algebraic equations. In principle it can straightforwardly be extended to partial differential equations, as already shown in [5]. But there are a number of problems making it rather difficult to formulate a rigorous general theory. This begins already with the question of what function spaces and norms should be used. We will not discuss these points but for simplicity restrict ourselves to smooth functions with the supremum norm. We do not expect serious changes, if more general spaces and norms, especially other L^p norms, are used.

Let $\mathcal{C}^\infty(D)$ denote the space of smooth $\mathbb{R}^m$-valued functions defined on some domain $D \subset \mathbb{R}^n$ and $\|\cdot\|$ the usual Euclidean norm in $\mathbb{R}^m$. For $f \in \mathcal{C}^\infty(D)$ we denote the supremum norm by $\| f \|_\infty = \sup_{x\in D} \| f(x) \|$. Furthermore we introduce on $\mathcal{C}^\infty(D)$ for $\ell \in \mathbb{N}$ the Sobolev type norms

$$\| f \|_\ell = \sum_{0\leq|\mu|\leq\ell} \| \frac{\partial^{|\mu|} f}{\partial x^\mu} \|_\infty \ . \tag{9}$$

Partial differential equations are usually accompanied by initial or boundary conditions. In order to accommodate for this we take the following simple approach. Let $D' \subset D$ be a subdomain, typically of lower dimension, and introduce on $\mathcal{C}^\infty(D')$ similar norms denoted by $\|\cdot\|'_\infty$ etc. The conditions are then written in the form $\Psi^\sigma(x,u,p)\big|_{D'} = 0$. This comprises most kinds of initial or boundary conditions. The highest derivative in Ψ^σ determines the order of the conditions.

Definition 4. Let $\Phi^\tau(x,u,p) = 0$ be a system of differential equations and let $U(x)$ be a smooth solution of it defined on some domain D and satisfying some initial/boundary conditions of order k on a domain $D' \subset D$. The system has the *perturbation index* $\nu_{\rm p}$ along this solution, if $\nu_{\rm p}$ is the smallest integer such that for solutions $\hat{U}(x)$ of the perturbed system $\Phi^\tau(x,u,p) = \delta^\tau(x)$ there exists an estimate

$$\| \hat{U}(x) - U(x) \| \leq C \left(\| \hat{U} - U \|'_k + \| \delta \|_{\nu_{\rm p}-1} \right) , \tag{10}$$

whenever the right hand side is sufficiently small. The constant C may depend only on the domains D, D' and on the functions Φ^τ.

In the case of first order differential algebraic equations this reduces to the definition of Hairer et al. [7]. D' contains then only the initial point x_0 and the initial conditions do not depend on derivatives; the norm $\|\cdot\|'_0$ is simply the Euclidean norm on $\mathbb{R}^m$. This definition is almost identical to the one proposed by Campbell and Marszalek [5]. The main difference is that we do not include the order k of the initial/boundary conditions into the index.

There exist several ways to interpret the functions δ^τ appearing in (10). Hairer et al. considered them mainly as *defect*, i.e. they assumed that the functions $\hat{U}$ are given and that entering them into the differential equations yields δ. As indicated by the phrasing of Def. 4 we take another point of view and consider δ^τ as a perturbation of the original differential equation.

This approach allows us immediately to generalise the definition of the perturbation index $\nu_{\rm p}$. Let $\tilde{\Phi}^\tau(x,u,p,\delta_1,\ldots,\delta_t)$ be some functions depending smoothly on the $\delta_j(x)$ and their derivatives up to some order $\bar{q}$ such that $\tilde{\Phi}^\tau(x,u,p,0,\ldots,0) = \Phi^\tau(x,u,p)$. Note that there are no restrictions on the number of perturbations δ_j, on $\bar{q}$ or on the way $\tilde{\Phi}^\tau$ depends on them. Now we can replace in Def. 4 the condition that $\hat{U}$ be a solution of $\Phi^\tau(x,u,p) = \delta^\tau$ by the condition that $\hat{U}$ be a solution of $\tilde{\Phi}^\tau(x,u,p,\delta) = 0$.

For arbitrary systems $\Phi^\tau = 0$ the perturbations δ_j cannot be chosen freely, as in general they must satisfy *compatibility conditions* in order to ensure that the system $\tilde{\Phi}^\tau(x,u,p,\delta) = 0$ has solutions: in the case of the trivial system $\dot{u} = 0$ and $u = 0$, the perturbed system $\dot{u} = \delta_1$, $u = \delta_2$ is solvable only when $\dot{\delta}_2 = \delta_1$.

Basically this is also the point of view taken by Campbell and Marszalek [5] who speak of *forcing functions* or *input* instead of perturbations. These terms are motivated by control theory and indicate that in principle it is at any time possible to change the meaning of u and δ, i.e. to consider δ as the dependent variables of the differential equation and u as perturbation. Obviously such changes in the interpretation of the equation will in general lead to different values of the perturbation index.

Thus formally it makes no problem to generalise the notion of a perturbation index to partial differential equations. However, it is a more delicate matter to decide whether this generalisation makes sense. For differential algebraic equations satisfying a Lipschitz condition it is straightforward to show (based on Gronwall's lemma) that for normal systems, i.e. explicit systems of the form $\dot{u} = \phi(t,u)$, we have $\nu_{\mathrm{P}} = 0$. In the case of partial differential equations this is less obvious.

For *linear* normal systems one may take a semigroup approach and consider the partial differential equation as an abstract ordinary differential equation on some function (Banach) space $\mathcal{B}$. Let the equation be $\dot{u} = Au$ where the operator A is the infinitesimal generator of a semigroup $T(t)$. It is a classical result (see e.g. [23]) that the solution $U(t)$ of the initial value problem $\dot{u} = Au + \delta(t)$, $u(0) = u_0$ with some $\mathcal{B}$ valued function δ satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\delta(s)\,ds\,. \tag{11}$$

If the operators $T(t)$ are bounded, this leads directly to an estimate of a form similar to Def. 4 for $\nu_{\mathrm{P}} = 0$.

Note that these simple considerations apply only for perturbed equations of the form $\Phi^\tau(x,u,p) = \delta^\tau$. If one uses more general perturbations $\tilde{\Phi}^\tau(x,u,p,\delta) = 0$, one must be careful even in the case of a differential algebraic equation, as the equation may not be structurally stable and thus completely change its properties under a perturbation.

4 Relations between the Indices

After having defined all these indices we exhibit now some relations between them. In the case of differentiation indices we restrict ourselves to some elementary consequences of formal theory. In Sect. 4.2 we discuss how an estimate for the perturbation index ν_{P} can be obtained as the involution index of the perturbed system.

4.1 Differentiation Indices

By definition any involutive system is formally integrable. But a formally integrable system requires in general some prolongations to become involutive. Of course no projections are needed, as all non-trivial integrability conditions are already included in the system. Thus we get $\nu_{\mathrm{f}} \leq \nu_{\mathrm{i}}$. An example where $\nu_{\mathrm{f}} < \nu_{\mathrm{i}}$ was given in Sect. 2.4. A similarly trivial result is the inequality already discussed above: $\nu_{\mathrm{d}} \leq \nu_{\mathrm{f}}$. Recall that even for differential algebraic equations it is possible to have here a strict inequality!

To find for partial differential equations an upper bound for the difference $\nu_{\mathrm{i}} - \nu_{\mathrm{f}}$ requires a fairly deep result from formal theory. We mentioned already in Sect. 2.3 the prolongation theorem stating that every symbol becomes involutive after a finite number of prolongations. A proof of it yielding an upper bound for the number of prolongations was given by Sweeny [32]. This bound depends only on the order q of the differential equation, the number n of independent and the number m of dependent variables and can be expressed in form of a recursion relation [18].

If we denote by $\hat{q}(n,m,q)$ the upper bound for the order at which the symbol $\mathcal{M}_q$ becomes involutive, then

$$\begin{aligned} \hat{q}(n,m,q) &= \hat{q}\left(n, m\binom{q+n-1}{n}, 1\right), \\ \hat{q}(n,m,1) &= m\binom{\hat{q}(n-1,m,1)+n}{n-1} + \hat{q}(n-1,m,1) + 1\,, \\ \hat{q}(0,m,1) &= 0\,. \end{aligned} \tag{12}$$

Obviously, $\hat{q}$ grows rapidly; e.g. $\hat{q}(4,4,1) = 13.151.182.504$. Fortunately, it usually drastically overestimates the number of prolongations actually needed. It is not known whether this bound is sharp, i.e. whether there really exist symbols where one needs such a huge number of prolongations.

Consider the following simple system where both equations are of order q

$$\mathcal{R}_q \;:\; \begin{cases} u_{x\cdots x} = 0\,, \\ u_{y\cdots y} = 0\,. \end{cases} \tag{13}$$

$\mathcal{R}_q$ is formally integrable but not involutive. One can show that its symbol becomes involutive after $q-1$ prolongations [29]. But evaluation of (12) yields $q^4/4 + q^3/2 + 9q^2/4 + 2q + 2$ for this system.

Note that this result yields only a bound for the difference $\nu_{\mathrm{i}} - \nu_{\mathrm{f}}$ and not on ν_{i}. This can be seen from the discussion of the completion algorithm in Sect. 2.3. The considerations above apply only to its inner loop. The termination of the outer loop was shown with a Noetherian argument giving no indications about the number of iterations needed.

For ordinary differential equations the situation is much simpler. As discussed in Sect. 2.5 they always possess an involutive symbol and we can skip the inner loop in our completion algorithm. This entails that the final involutive system is of the same order as the original system and it is straightforward to give an upper bound: $\nu_{\mathrm{f}} = \nu_{\mathrm{i}} \leq \dim \mathcal{R}_q$. Namely, in the worst case we add one integrability condition in each iteration of the outer loop. But as soon as we have added $\dim \mathcal{R}_q$ equations, we obtain a system with a zero-dimensional solution space. Such a system must be either formally integrable or inconsistent, as no more independent equations can be added.

4.2 Perturbation Indices

In general it is rather difficult to obtain estimates like the one required by Def. 4 of the perturbation index ν_{P}. Thus there is much interest in relating ν_{P} with differentiation indices which we can (at least in principle) determine algorithmically. We follow here a slightly different route: we try to bound ν_{P} by the involution index not of $\mathcal{R}_q$ but of the perturbed equation $\tilde{\mathcal{R}}_q$.

We begin by considering linear differential equations. Let $\Phi^\tau(x, u, p) = 0$ be a local representation of the equation $\mathcal{R}_q$. As in Sect. 3.3 we associate to this local representation the perturbed equation $\tilde{\mathcal{R}}_q$ defined locally by $\Phi^\tau(x, u, p) = \delta^\tau(x)$. Now we can apply the completion algorithm of Sect. 2.3. Since we are dealing with a linear system, this algorithm will require only linear operations. Hence the finally obtained involutive equation $\tilde{\mathcal{R}}^{(\tilde{s})}_{q+\tilde{r}}$ possesses a local representation of the form $\psi^\sigma(x, u, p) = F^\sigma(x, \delta, \partial_x\delta, \ldots)$. If some of the ψ^σ vanish, we have found compatibility conditions $F^\sigma(x, \delta, \partial_x\delta, \ldots) = 0$ which the perturbations must satisfy. So these are automatically determined during the completion of $\tilde{\mathcal{R}}_q$.

Let $\tilde{\nu}_{\mathrm{i}} = \tilde{r} + \tilde{s}$ be the involution index of $\tilde{\mathcal{R}}_q$. As we performed $\tilde{\nu}_{\mathrm{i}}$ prolongations during the completion, in the right hand sides F^σ no derivatives of δ of order higher than $\tilde{\nu}_{\mathrm{i}}$ appear. Now we can apply the ideas used in the proof of the Cartan-Kähler theorem outlined in Sect. 2.3. According to these we can construct the solutions of $\tilde{\mathcal{R}}^{(\tilde{s})}_{q+\tilde{r}}$ by solving a sequence of normal subsystems using the results of one step as initial data for the next step.

At each step the solution can be expressed in the form (11), i. e. an integral over the right hand side appears. Collecting all these expressions we can deduce an estimate of the form (10) with $\nu_{\mathrm{P}} \leq \tilde{\nu}_{\mathrm{i}}$. ν_{P} will be less than $\tilde{\nu}_{\mathrm{i}}$, if the highest order derivatives of δ appear only in the compatibility conditions and not as right hand sides of equations involving the unknown functions u. In any case we can conclude that the involution index $\tilde{\nu}_{\mathrm{i}}$ of the perturbed equation $\tilde{\mathcal{R}}_q$ represents an upper bound for the perturbation index ν_{P} of the original equation $\mathcal{R}_q$.

For nonlinear equations or more general forms of perturbations the situation is not so clear. But this is not surprising, as in these cases we could not even show that the perturbation index of normal systems generally vanishes. Nevertheless $\tilde{\nu}_{\mathrm{i}}$ should at least give some indications about the size of ν_{P}. As in the linear case the involutive equation $\tilde{\mathcal{R}}^{(\tilde{s})}_{q+\tilde{r}}$ will depend on derivatives of δ up to order $\tilde{\nu}_{\mathrm{i}}$ and it is therefore only reasonable to assume that in an estimate of the form (10), these derivatives should appear.

So far we have neglected the fact that in Def. 4 ν_{p} was defined along a specific solution $\hat{U}$ and of course ν_{p} may take different values along different solutions. For this reason Campbell and Gear [3] introduced the *maximum perturbation index* ν_{mp} as the maximal value of ν_{p} on the solution space of the differential equation. Obviously $\tilde{\nu}_{\mathrm{i}}$ represents an upper bound of ν_{mp}, as the arguments above are independent of the choice of a specific solution $\hat{U}$.

5 Conclusions

We have shown in this article how many of the index concepts appearing in the literature on differential algebraic equations find a simple geometric interpretation within the framework of the formal theory. The first to point out this connection and to apply it to concrete numerical problems were Le Vey and Tuomela (see [14] and references therein).

This new interpretation is interesting for several reasons. The perhaps most important one is that it allows us to extend the theory to general systems of partial differential equations. We also think that it provides useful insights. For example, it emphasises that in such systems the main problems typically come from equations of lower class and not of lower order. This is a completely new phenomenon not present in differential algebraic equations. For this reason we do not think that the sometimes found terminology "partial differential algebraic equation" (PDAE) is a very good one.

We believe that it will be the main challenge in the numerical analysis of such general systems to develop robust methods to handle this situation which appears naturally in a number of important applications (we mentioned already electrodynamics, but the same holds e. g. for the dynamics of incompressible fluids). Two simple approaches are either to use an underlying equation or to follow the ideas of the method of lines.

In the first case one follows directly the strategy of the Cartan-Kähler theorem and considers equations of lower class as constraints on the initial data for the equations of maximal class which are taken as underlying equation. However, in general one will have to take measures against a numerical "drift off the constraints." Further problems like spurious solutions arise, if one imposes not only initial but also boundary conditions, as it is usually the case. A thorough discussion for the special case of Maxwell's equations can be found in [10].

In a method of lines approach one would keep the derivatives with respect to x^n and discretise all other derivatives. If equations of lower class are present, they will become purely algebraic equations; thus we arrive naturally at a differential algebraic equation. For semi-linear systems of the form $Au_t + Bu_{xx} + F(u) = f(t,x)$ where at least one of the matrices A, B is singular, this approach has been studied in detail in [15–17].

Actually, most attempts to define (differentiation) indices for partial differential equations that can be found in the literature are based on such semi-discretisations (or closely related integral transforms). This is in marked contrast to differential algebraic equations where indices are defined without reference to a numerical method. The same holds for our generalisation to partial differential equations which are furthermore defined for completely arbitrary systems and not only special classes.

Discretisation based indices cannot capture general structural properties of a given differential equation. But it should be very useful to compare them with discretisation independent indices like the ones introduced in this article (some steps in this direction can already be found in [5]). Intuitively one would want that both indices coincide. If the discretisation based index was higher, this could indicate that unnecessary complications have been introduced. Conversely, if it was lower, one might have lost some important features of the differential equation during the discretisation. Of course, this requires that one develops comparable indices.

Another useful insight provided by formal theory concerns an important difference between differentiation and perturbation indices. All of our differentiation indices are defined *intrinsically*, i.e. they depend only on the manifold $\mathcal{R}_q$ and not on any specific local representation $\Phi^\tau(x,u,p) = 0$. Hence they remain invariant under algebraic manipulation of the equations or under changes of coordinates; they really capture structural properties of the differential equation.

In contrast, the perturbation index ν_p will in general have a different value after algebraic manipulations or changes of coordinates. The same holds for our estimate $\tilde{\nu}_\mathrm{f}$. Of course, $\tilde{\nu}_\mathrm{f}$ is defined intrinsically for $\tilde{\mathcal{R}}_q$, however the definition of this submanifold is not intrinsic but depends on the chosen representation $\Phi^\tau(x,u,p) = 0$ of $\mathcal{R}_q$.

Finally, it should be remarked that we made implicitly a number of assumptions in this article. Most notably we had to assume that our equations are sufficiently smooth, so that our completion algorithm can be performed; equations with discontinuous terms should be treated with much care. We omitted any discussion of branchings in the completion algorithm. In general, involution is a local property and index values may vary on the manifold $\mathcal{R}_q$. We also ignored the problem of δ-regularity which makes the formal analysis much more complicated, if characteristic coordinates are used.

All our solvability results apply only to analytic systems and solutions. This is of course much to restrictive for applications and due to the fact that the only general existence and uniqueness theorem for normal systems of partial differential equations is the Cauchy-Kowalevsky theorem requiring analyticity. However, the step by step technique of the proof of the Cartan-Kähler theorem can also be applied in other situations, if more general results are available. For example, in [25,28] the uniqueness theorem of Holmgren was generalised from normal linear systems to arbitrary involutive ones. Analogously one could prove much stronger existence and uniqueness theorems for Maxwell's equations (or other equations with similar structure) based on classical theorems for special classes of normal systems.

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