Perturbation versus Differentiation Indices*

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Abstract. We discuss the relation between perturbation and differentiation indices for overdetermined systems of differential equations based on the formal theory. We show how the Cartan normal form of an involutive system can be used to extend the notion of an underlying equation from differential algebraic equations to partial differential equations. This allows us to generalise results of Campbell and Gear on the relation between perturbation and differentiation indices to systems of partial differential equations.

1 Introduction

This article is a sequel to [17] and further elaborates and rigorously proves some statements given there. We are especially concerned with the relation of two different kinds of index concepts for overdetermined systems of differential equations. *Differentiation indices* essentially count the number of differentiations required until one obtains a system with certain properties. This makes them comparatively easy to compute and thus they are much used. On the other hand, one may wonder why they should give any indication of difficulties appearing in the numerical integration of such systems. This becomes much more evident with *perturbation indices* based on estimates of the difference between solutions of the exact and a perturbed system. But such estimates are often rather hard to establish. For this reason it is of interest to relate differentiation and perturbation indices.

For the case of overdetermined systems of ordinary differential equations, often called *differential algebraic equations* [2,6], this problem has eventually been solved by Campbell and Gear [4]. The main purpose of this article is to extend their work to non-normal systems of partial differential equations. For this we first review their results in a more geometric manner that clarifies the main ingredients needed; in particular the important role of underlying equations is emphasised. Then we show how these ingredients can be defined for partial differential equations using the so-called Cartan normal form of an involutive system. This allows us, at least for linear systems, to extend their proof straightforwardly.

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2 Geometry of Differential Algebraic Equations

Most approaches to differential algebraic equations are purely computational but some geometric ones [12,13] have been developed, too. Although these are typically a bit more restricted in their applicability (often they implicitly make some constant rank assumptions), they have the advantage to exhibit more clearly the basic ideas. For this reason we will mainly work geometrically in this article. We will furthermore assume in this section that we are dealing with autonomous systems; this is only for simplifying the presentation and not a real restriction.

Given a normal system of ordinary differential equations $\boldsymbol{y}' = \boldsymbol{Y}(\boldsymbol{y})$, its geometric counterpart is a vector field $\boldsymbol{Y}$ living on some manifold M (often simply $\mathbb{R}^m$). A differential algebraic equation is a more complicated object. Instead of a single vector field we are now dealing with a whole *pencil* of vector fields on the ambient manifold M and in addition we are given a submanifold $N_0 \subset M$ (see Fig. 1).

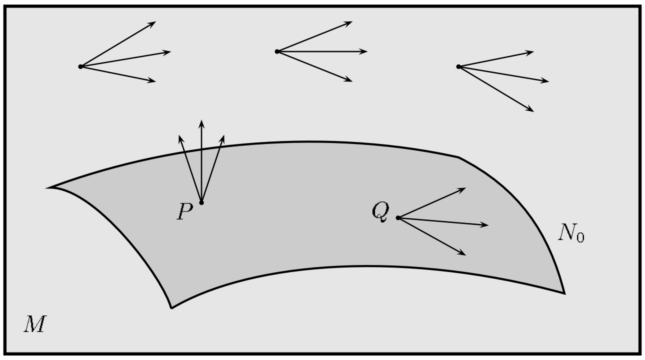

Fig. 1. Vector field pencil and submanifold

Example 1. The pencil and the submanifold are easily read off, if the differential algebraic equation is given in semi-explicit form. As a trivial example we consider the linear equation $y_2' = y_1 - 2$, $y_3' = -y_2$ and $y_2 = 3$. Here $M = \mathbb{R}^3$ and the submanifold N_0 is the plane described by the constraint equation $y_2 = 3$. We have a pencil of vector fields, as we do not know the y_1-component. The fields contained in the pencil can be written in parametric form as $\boldsymbol{Y}_a(\boldsymbol{y}) = (a(\boldsymbol{y}), y_1 - 2, -y_2)^t$ where a is an arbitrary function. $\lhd$

Given a differential algebraic equation it is not at all clear that it actually possesses solutions and if yes where they live. For this reason one must first *complete* the system to a *formally integrable* or *involutive* one. One may also consider this operation as a consistency check. In our geometric framework it amounts to imposing *tangency conditions*.

Any solution of the differential algebraic equation must satisfy two conditions: it must define an integral curve for a vector field in the pencil and this curve must lie in the submanifold N_0. But these two conditions can be fulfilled simultaneously only, if at least some vector fields in the pencil are tangent to the submanifold N_0.

As one can see in Fig. 1, we must distinguish two cases. If all vector fields in the pencil are transversal to N_0 at some point $P \in N_0$, then no integral curve through P can lie in N_0 and we must eliminate this point. These eliminations lead to a smaller submanifold $N_1 \subset N_0$. From a computational point of view, this means that there are some "hidden" *algebraic* constraints. The second case arises, if the pencil contains at a point $Q \in N_0$ vector fields transversal to N_0. Then we must remove these and shrink the pencil to the tangential fields. In concrete computations this means that differentiating the constraints yields new *differential* equations.

After having imposed the first tangency condition, we deal in general with a smaller submanifold $N_1 \subseteq N_0$ and a shrunk vector field pencil. In case that N_1 is a proper submanifold of N_0, we must again analyse the tangency of the pencil, but this time to the *smaller* submanifold N_1. This might lead to a yet smaller submanifold N_2 and a further shrinking of the pencil.

After a finite number of steps we reach a final submanifold[1] N_k and a vector field pencil where all contained fields are tangent to N_k. This is called an *involutive* system. In most applications, the final pencil consists only of a single vector field as shown in Fig. 2. Otherwise we are dealing with an underdetermined system. The final submanifold consists of all points which may be used as initial values for the given differential algebraic equation.

If the final pencil contains only a single vector field, it is often called an *underlying* field for the differential algebraic equation. One should note that such a field is uniquely defined only *on* the final submanifold N_k. If we add to an underlying vector field any field that vanishes on N_k, we obtain again an underlying field: for the differential algebraic equation it is only of relevance what happens on N_k.

Example 2. In our simple example above, the completion needs two steps. In the first step we require that the vector fields $\boldsymbol{Y}_a$ are tangent to the plane N_0 or equivalently that the y_2-component vanishes on N_0. This cannot be achieved by shrinking the pencil, i. e. by choosing a special form of the function $a(\boldsymbol{y})$, but only by restricting to the straight line $N_1 \subset N_0$ described by the two equations $y_2 = 3$ and $y_1 = 2$.

In the second step we check whether all vector fields $\boldsymbol{Y}_a$ in the pencil are tangential to N_1. Obviously, this requires that the y_1-component must vanish on N_1. This time this can be achieved by shrinking the pencil, namely

[1] If the system is not consistent, i. e. if it does not possess any solutions, then we obtain at some stage an empty submanifold.

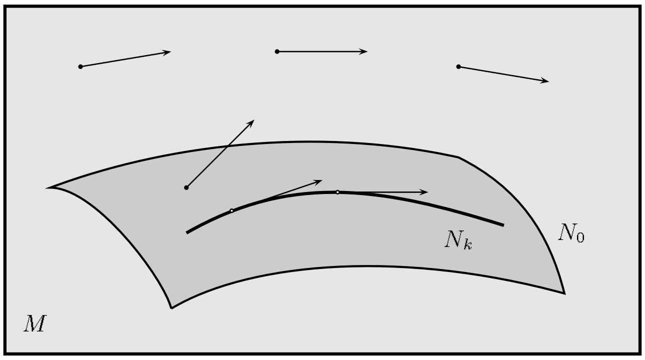

Fig. 2. Completed system

by setting $a(\boldsymbol{y}) = 0$. As $N_2 = N_1$, the completion process stops here. The final submanifold is the straight line N_1 and the pencil has shrunk to a single underlying vector field, e. g. $\boldsymbol{Y} = (0, 0, -3)^t$. More generally, any vector field on $\mathbb{R}^3$ that takes this form restricted to N_1 is an underlying field. We could also choose $\boldsymbol{Y} = (y_1 - 2, y_2 - 3, y_1 - y_2 - 2)^t$. $\triangleleft$

Note an obvious but very important property of the final constraint manifold N_k: it is an *invariant manifold* for the flow of any underlying vector field $\boldsymbol{Y}$: if $\phi(\boldsymbol{y}) = 0$ belongs to the equations describing N_k, its derivative along flow lines, $\dot{\phi} = \boldsymbol{Y}\phi$, vanishes on N_k. Thus ϕ is a *weak invariant*[2].

A simple approach to the numerical integration of a differential algebraic equation is to take an underlying vector field and to compute its integral curve through a point on N_k. Because of the weak invariance, the integral curve will completely lie on N_k. Unfortunately, this does not necessarily hold for its numerical approximations; in general, we must expect a *drift* off N_k.

How severe this drift is depends on the stability properties of the invariant manifold N_k under the flow of the chosen underlying vector field. Here the non-uniqueness of underlying vector fields off N_k is crucial; different extensions will possess different stability properties. For the special case of constrained Hamiltonian systems a comparison of different underlying fields can be found in [16,18].

3 Differentiation and Perturbation Indices

As already indicated in [17], the basic idea behind *differentiation indices* is to count the number of tangency conditions that have to be imposed until a certain state is reached. Looking at our geometric description in the last section, we see that there are two natural choices.

[2] It is not a first integral, as generally $\boldsymbol{Y}\phi$ does not vanish everywhere on M.

Definition 3. The *involution index* ν_i is the minimal number of tangency conditions that must be imposed until an involutive system is reached, i. e. the final submanifold is N_{ν_i}.

The involution index is defined for any differential algebraic equation, as any system can be completed in a finite number of steps. It has been introduced by several authors under different names. It is for example contained in the geometric works [12,13] and it is equivalent to the strangeness index of Kunkel and Mehrmann [8,9].

Definition 4. The *determinacy index* ν_d is the minimal number of tangency conditions that must be imposed until the pencil has shrunk to a single vector field, i. e. until an underlying vector field has been found.

The determinacy index is *not* always defined: for underdetermined systems no underlying vector field exists; this property explains the name. It is equivalent to "the" differentiation index, the most used index concept for differential algebraic equations. It is usually defined via so-called derivative arrays, but these represent just a computational approach to the geometric tangency conditions we impose.

In our example in the last section we have $\nu_d = \nu_i = 2$. In general, we have obviously $\nu_d \leq \nu_i$. An existence theory can only be built upon the involution index, as only after the full completion we know whether the given differential algebraic equation is consistent. So one may wonder about the importance of the determinacy index. We will answer this question in the next section.

The notion of a *perturbation index* was introduced by Hairer, Lubich and Roche [5]. It is a purely analytic concept based on estimating the effect of perturbing a given differential algebraic equation and measures up to which order derivatives of the perturbation enter the estimate.

Definition 5. Let $\boldsymbol{y}(x)$ be a solution of the implicit differential algebraic equation[3] $\boldsymbol{F}(x,\boldsymbol{y},\boldsymbol{y}') = 0$ and $\hat{\boldsymbol{y}}(x)$ a solution of the perturbed equation $\boldsymbol{F}(x,\hat{\boldsymbol{y}},\hat{\boldsymbol{y}}') = \boldsymbol{\delta}(x)$ where $\boldsymbol{\delta}(x)$ is a sufficiently often differentiable function. Assuming that both solutions are defined on the interval $[0,X]$, we say that the equation $\boldsymbol{F} = 0$ has *perturbation index* ν_p along the solution $\boldsymbol{y}(x)$, if ν_p is the smallest integer such that the following estimate holds for all $x \in [0,X]$ provided the right hand side is sufficiently small:

$$|\boldsymbol{y}(x) - \hat{\boldsymbol{y}}(x)| \leq C\left(|\boldsymbol{y}(0) - \hat{\boldsymbol{y}}(0)| + \|\boldsymbol{\delta}\|_{\nu_p-1}\right) . \tag{1}$$

Here the constant C may depend only on $\boldsymbol{F}$ and the length X of the interval. $|\cdot|$ denotes some vector norm on $\mathbb{R}^m$ and $\|\cdot\|_k$ is a kind of Sobolev norm $\|\boldsymbol{\delta}\|_k = \sum_{i=0}^{k} \left\|\boldsymbol{\delta}^{(i)}\right\|$ based on a norm $\|\cdot\|$ on $\mathcal{C}([0,X],\mathbb{R}^t)$ which is usually either the maximum norm $\|\cdot\|_{L^\infty}$ or the uniform norm $\|\cdot\|_{L^1}$.

[3] We do not require here that there are as many equations as unknowns; the values of $\boldsymbol{y}$ are m-dimensional vectors, those of $\boldsymbol{F}$ are t-dimensional.

Whereas it is not so clear why the differentiation indices should indicate the degree of difficulty of the numerical integration of a differential algebraic equation, this is rather obvious for the perturbation index. If we take for $\hat{\boldsymbol{y}}$ an approximate solution of $\boldsymbol{F} = 0$, we may interpret the perturbation $\boldsymbol{\delta}$ as the residual obtained by entering it into the equation. (1) tells us that for an equation with $\nu_p > 1$ it does not suffice to keep this residual as small as possible, as also some of its derivatives enter the estimate. Strictly speaking, this implies that the initial value problem for such an equation is ill-posed.

In general there is no relation between ν_d, ν_i and ν_p, as demonstrated with some examples by Campbell and Gear [4]. This is not so surprising, as the perturbation index is based on a perturbed equation and the perturbation affects for non-linear equations the completion.

Example 6. Consider the following simple quasi-linear system [4]:

$$y_2 y_2' + y_1 = 0\,, \quad y_2 = 0\,. \tag{2}$$

Obviously this system is equivalent to $y_1 = y_2 = 0$. Thus its indices are $\nu_d = \nu_i = 1$, as after one step we obtain an underlying equation $y_1' = y_2' = 0$.

For the perturbed system

$$y_2 y_2' + y_1 = \delta_1\,, \quad y_2 = \delta_2 \tag{3}$$

the situation is different. The first tangency condition yields $y_2' = \delta_2'$ and then the first equation of (3) is equivalent to $y_1 = \delta_1 - \delta_2\delta_2'$. Imposing a second tangency condition leads to $y_1' = \delta_1' - (\delta_2\delta_2')'$ and thus an underlying equation. The indices of the perturbed system are $\tilde{\nu}_d = \tilde{\nu}_i = 2$.

As shown in [4] this example can be extended. Using an m-dimensional vector of unknowns $\boldsymbol{y}$, one can construct systems such that for the unperturbed system still $\nu_d = \nu_i = 1$ but for the perturbed one $\tilde{\nu}_d = \tilde{\nu}_i = m$. Thus the difference between the unperturbed and the perturbed indices may become arbitrarily large. $\triangleleft$

Definition 7. Let $\boldsymbol{F}(x, \boldsymbol{y}, \boldsymbol{y}') = 0$ be a differential algebraic equation. Its *perturbed determinacy index* $\tilde{\nu}_d$ and its *perturbed involution index* $\tilde{\nu}_i$, respectively, are defined as the indices of the perturbed differential algebraic equation $\boldsymbol{F}(x, \boldsymbol{y}, \boldsymbol{y}') = \boldsymbol{\delta}(x)$.

This definition implicitly assumes that the completion process of the perturbed equation is independent of the precise form of $\boldsymbol{\delta}$. In most applications this is probably the case. Otherwise, the perturbed indices are computed with the generic branch of the completion, i.e. under the assumption that the perturbation does *not* satisfy any special conditions.

4 Estimating the Perturbation Index

We have seen in Example 6 that the indices ν_d, ν_i and ν_p, respectively, are in general not related. It is now a natural thought to search for a relation between $\tilde{\nu}_d$, $\tilde{\nu}_i$ and ν_p, as all these indices are based on the perturbed system. But before we recall some auxiliary results.

Lemma 8 (Gronwall). *Let $I = [a,b] \subseteq \mathbb{R}$ and let $\phi : I \to \mathbb{R}$ be a continuous function. Suppose that on I*

$$\phi(x) \leq A + B \int_a^x \phi(\xi) d\xi \tag{4}$$

for some constants $A, B \geq 0$. Then for all $x \in I$

$$\phi(x) \leq A e^{B(x-a)} . \tag{5}$$

This is a classical result that can be found with proof in almost any textbook on ordinary differential equations (see e.g. [22]). Based on it we can compute ν_p for normal systems of ordinary differential equations.

Proposition 9. *Let $\boldsymbol{F}(x, \boldsymbol{y}, \boldsymbol{y}') = 0$ be an implicit ordinary differential equation such that the Jacobian $\partial \boldsymbol{F}/\partial \boldsymbol{y}'$ has a bounded inverse on $[0, X]$. Furthermore let $\boldsymbol{F}$ be Lipschitz continuous in $\boldsymbol{y}$ with constant L_0 and let $\hat{\boldsymbol{y}}(x)$ be a solution of the perturbed equation $\boldsymbol{F}(x, \hat{\boldsymbol{y}}, \hat{\boldsymbol{y}}') = \boldsymbol{\delta}(x)$. Then for all $x \in [0, X]$*

$$|\boldsymbol{y}(x) - \hat{\boldsymbol{y}}(x)| \leq C \left(|\boldsymbol{y}(0) - \hat{\boldsymbol{y}}(0)| + \int_0^x |\boldsymbol{\delta}(\xi)| \, d\xi \right) \tag{6}$$

where the constant C depends only on $\boldsymbol{F}$ and the length X of the interval.

Proof. We solve the original and the perturbed equation for the derivatives $\boldsymbol{y}'$ and $\hat{\boldsymbol{y}}'$, respectively, which is always possible under the made assumptions. This yields the equations $\hat{\boldsymbol{y}}' = \boldsymbol{f}(x, \hat{\boldsymbol{y}}, \boldsymbol{\delta})$ and $\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y}, 0)$. The function $\boldsymbol{f}$ satisfies a Lipschitz condition in both $\hat{\boldsymbol{y}}$ and $\boldsymbol{\delta}$ with a constant L_1 depending on L_0 and the bound of $(\partial \boldsymbol{F}/\partial \boldsymbol{y}')^{-1}$. Now we integrate both equations from 0 to x, subtract them and take the norm:

$$|\boldsymbol{y}(x) - \hat{\boldsymbol{y}}(x)| \leq |\boldsymbol{y}(0) - \hat{\boldsymbol{y}}(0)| + \left| \int_0^x \left[\boldsymbol{f}\left(\xi, \boldsymbol{y}(\xi), 0\right) - \boldsymbol{f}\left(\xi, \hat{\boldsymbol{y}}(\xi), \boldsymbol{\delta}(\xi)\right) \right] d\xi \right| . \tag{7}$$

Interchanging the integration and the norm, we can exploit the Lipschitz continuity of $\boldsymbol{f}$ to obtain

$$|\boldsymbol{y}(x) - \hat{\boldsymbol{y}}(x)| \leq |\boldsymbol{y}(0) - \hat{\boldsymbol{y}}(0)| + L_1 \int_0^x \Big(|\boldsymbol{y}(\xi) - \hat{\boldsymbol{y}}(\xi)| + |\boldsymbol{\delta}(\xi)| \Big) d\xi . \tag{8}$$

Obviously, $\int_0^x |\boldsymbol{\delta}(\xi)| \, d\xi$ is a monotonically increasing function of x on the interval $[0, X]$ taking its maximal value at $x = X$. Applying Gronwall's Lemma with $\phi(x) = |\boldsymbol{y}(x) - \hat{\boldsymbol{y}}(x)|$ and $A = |\boldsymbol{y}(0) - \hat{\boldsymbol{y}}(0)| + L_1 \int_0^X |\boldsymbol{\delta}(\xi)| \, d\xi$, $B = L_1$ yields immediately the claimed estimate with $C = \max(1, L_1) \exp(L_1 X)$. $\square$

Thus a fully implicit system has generally the perturbation index $\nu_p = 1$, as the right hand side of (6) is the L^1 norm of $\boldsymbol{\delta}$. But note that in order to derive (8) we had to interchange in (7) the integration and the norm. If we start with an *explicit* system, so that the perturbed form is $\hat{\boldsymbol{y}}' = \boldsymbol{f}(x, \hat{\boldsymbol{y}}) + \boldsymbol{\delta}$, such an interchange is not necessary and we obtain on the right hand side of (6) a term of the form $\left|\int_0^x \boldsymbol{\delta}(\xi)d\xi\right|$. Interpreting $\|\boldsymbol{\delta}\|_{-1}$ as the (maximum) norm of $\int_0^x \boldsymbol{\delta}(\xi)d\xi$, we see that such a system has in fact the perturbation index $\nu_p = 0$.

Theorem 10. *Let $\boldsymbol{F}(x, \boldsymbol{y}, \boldsymbol{y}') = 0$ be a differential algebraic equation. Then*

$$\tilde{\nu}_d \leq \nu_p \leq \tilde{\nu}_d + 1\,. \tag{9}$$

Proof. The essential step of the proof is to show that the completion of the perturbed form of the given differential algebraic equation leads to an underlying equation of the form

$$\boldsymbol{F}^c(x, \hat{\boldsymbol{y}}, \hat{\boldsymbol{y}}', \boldsymbol{\delta}, \boldsymbol{\delta}', \ldots, \boldsymbol{\delta}^{(\tilde{\nu}_d)}) = 0 \tag{10}$$

with $\partial\boldsymbol{F}^c/\partial\boldsymbol{\delta}^{(\tilde{\nu}_d)} \neq 0$. Then we can apply the proposition above and get an estimate of the form (1) with $\nu_p - 1 = \tilde{\nu}_d$. The lower bound is obtained by taking into account the remark above: if the integration can be left within the norm, it kills one derivative. ν_p cannot be less than $\tilde{\nu}_d$, as the solution of (10) depends at least on $\boldsymbol{\delta}^{(\tilde{\nu}_d - 1)}$, and by definition $\tilde{\nu}_d$ is the smallest number of differentiations leading to an underlying equation.

Before the completion we transform the perturbed differential algebraic equation into semi-explicit form; this is a purely algebraic process which does not introduce any derivatives. As outlined in Section 2, the completion process takes then the following simple form: the algebraic equations are differentiated and it is checked whether this yields any new equations; the completion stops as soon as no new *algebraic* equations arise.

In the first step the algebraic part of the differential algebraic equation is differentiated. If this leads to any new equations, they contain a first order derivative of a perturbation. If some of the new equations are algebraic, they are differentiated again. Thus any new equations arising out of them contains a second order derivative of a perturbation and so on. The determinacy index $\tilde{\nu}_d$ counts the last completion step where new *differential* equations appear, i.e. equations that are part of the underlying equation $\boldsymbol{F}^c = 0$. Thus these equations depend effectively on derivatives of order $\tilde{\nu}_d$ of the perturbations and $\boldsymbol{F}^c$ is of the form claimed above. □

Example 11. We consider again the system of Example 6. From its closed form solution one can easily see that $\nu_p = 2$. Thus $\nu_p = \tilde{\nu}_d$ in agreement with our theorem. We are at the lower bound, as the underlying equation is of the *explicit* form $y_1' = \delta_1' - (\delta_2\delta_2')'$, $y_2' = \delta_2'$ and we can thus perform the integration within the norm. ◁

5 Involutive Systems of Partial Differential Equations

The formal theory of differential equations provides an intrinsic approach to very general systems of partial differential equations. In particular, it is able to handle effectively under- and overdetermined systems based on the notion of *involution*. This is now a considerably more complicated concept as in the case of ordinary differential equations comprising a mixture of geometric and algebraic ideas. For lack of space it is not possible to discuss here the formal theory in detail. We must refer to the literature [11,15]; some basic notions can also be found in [17]. A simple introduction into a combinatorial approach to involution is given in [3]; a deeper analysis is contained in [19].

Although we will describe everything in local coordinates, one should note that the formal theory is completely intrinsic and can be presented without any coordinates in a purely geometric manner. However, this requires the introduction of a certain machinery like jet bundles which we want to avoid here. For simplicity, we furthermore assume that we are given a first order system for m unknown functions u_α in n independent variables x_i without any algebraic equations.

For the analysis of differential algebraic equations it is very convenient to have them in semi-explicit form. Its analogue for systems of partial differential equations is the following triangular form where the numbers $\beta^{(i)}$ satisfy $0 \leq \beta^{(1)} \leq \cdots \leq \beta^{(n)} \leq m$.

$$
\begin{aligned}
\frac{\partial u_\alpha}{\partial x_n} &= \phi_{n\alpha}\left(\boldsymbol{x}, \boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial x_j}, \frac{\partial u_\delta}{\partial x_n}\right) , && \begin{cases} 1\leq\alpha\leq\beta^{(n)} , \\ 1\leq j<n , \\ \beta^{(n)}<\delta\leq m , \end{cases} \\
\frac{\partial u_\alpha}{\partial x_{n-1}} &= \phi_{n-1,\alpha}\left(\boldsymbol{x}, \boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial x_j}, \frac{\partial u_\delta}{\partial x_{n-1}}\right) , && \begin{cases} 1\leq\alpha\leq\beta^{(n-1)} , \\ 1\leq j<n-1 , \\ \beta^{(n-1)}<\delta\leq m , \end{cases} \\
&\vdots \\
\frac{\partial u_\alpha}{\partial x_1} &= \phi_{1\alpha}\left(\boldsymbol{x}, \boldsymbol{u}, \frac{\partial u_\delta}{\partial x_1}\right) , && \begin{cases} 1\leq\alpha\leq\beta^{(1)} , \\ \beta^{(1)}<\delta\leq m . \end{cases}
\end{aligned}
\tag{11}
$$

Here as many equations as possible have been solved for an x_n-derivative. Of the remaining ones as many as possible have been solved for an x_{n-1}-derivative and so on. The classical case of a system in Cauchy–Kovalevskaya form arises, if $\beta^{(n)} = m$ and all other $\beta^{(i)}$ vanish.

We say that those equations in system (11) which are solved for an x_k-derivative are of *class* k and assign them the *multiplicative variables* $x_1, \dots, x_k$. The system is *involutive*, if the following property holds: any equation obtained by differentiating one of the equations in (11) with respect to a non-multiplicative variable can be written as a linear combination of the equations in (11) plus those equations obtained by differentiating them with respect to their multiplicative variables only. For an involutive system, (11) is also called its *Cartan normal form*.

Example 12. Consider the following linear system with constant coefficients in three independent variables:

$$\boldsymbol{u}_{x_3} = A_1\boldsymbol{u}_{x_1} + A_2\boldsymbol{u}_{x_2}\,, \tag{12}$$

$$0 = B_1\boldsymbol{u}_{x_1} + B_2\boldsymbol{u}_{x_2}\,. \tag{13}$$

Here $A_1, A_2 \in \mathbb{R}^{m\times m}$ and $B_1, B_2 \in \mathbb{R}^{r\times m}$ for some $r \leq m$. The system is already in triangular form (though not explicitly solved for the x_2-derivatives) provided that the rank of B_2 is r what we assume. The equations in (12) are all of class 3 and thus all independent variables are multiplicative for them. The equations in (13) are only of class 2 and hence x_3 is non-multiplicative for them. There are no equations of class 1.

In order to check whether or not the system is involutive, we must analyse what happens, if we differentiate the equations in (13) with respect to x_3. This yields the equations $B_1\boldsymbol{u}_{x_1x_3} + B_2\boldsymbol{u}_{x_2x_3} = 0$. The second derivatives can be eliminated by subtracting x_1- and x_2-differentiations of the equations in (12) and we obtain $B_1A_1\boldsymbol{u}_{x_1x_1} + (B_1A_2 + B_2A_1)\boldsymbol{u}_{x_1x_2} + B_2A_2\boldsymbol{u}_{x_2x_2} = 0$. The system is involutive, if and only if these equations can be written as linear combinations of x_1- and x_2-differentiations of the equations in (13), i.e. if and only if two matrices $M_1, M_2 \in \mathbb{R}^{r\times m}$ exist such that $B_1A_1 = M_1B_1$, $B_2A_2 = M_2B_2$ and $M_1B_2 + M_2B_1 = B_1A_2 + B_2A_1$. If this is the case, we can write

$$\begin{aligned}\frac{\partial}{\partial x_3}\left(B_1\boldsymbol{u}_{x_1} + B_2\boldsymbol{u}_{x_2}\right) = {}& \left[B_1\frac{\partial}{\partial x_1} + B_2\frac{\partial}{\partial x_2}\right]\left(\boldsymbol{u}_{x_3} - A_1\boldsymbol{u}_{x_1} + A_2\boldsymbol{u}_{x_2}\right) - \\ & \left[M_1\frac{\partial}{\partial x_1} + M_2\frac{\partial}{\partial x_2}\right]\left(B_1\boldsymbol{u}_{x_1} + B_2\boldsymbol{u}_{x_2}\right).\end{aligned} \tag{14}$$

Thus we expressed the non-multiplicative differentiation on the left hand side as a linear combination of multiplicative ones. ◁

The *Cartan-Kähler Theorem* states that the (non-characteristic) initial value problem for an analytic involutive system with analytic initial data possesses a unique analytic solution. Its proof is based on the Cartan normal form and an iterative application of the classical Cauchy–Kovalevskaya Theorem to the subsystems of the normal form.

The *Cartan–Kuranishi Theorem* asserts that (under some regularity assumptions) any system of partial differential equations can be completed to an equivalent involutive one. Its proof is constructive. The arising completion method is *not* a straightforward generalisation of the simple procedure for differential algebraic equations outlined in Section 2. An efficient algorithmic realisation of it was derived in [7] combining geometric and algebraic ideas. It has been implemented in the computer algebra system *MuPAD* [1].

6 Indices for Partial Differential Equations

It is straightforward to extend the involution index to partial differential equations based on the Cartan-Kuranishi Theorem. Essentially, it requires only a reformulation of the definition above such that it can be applied to both ordinary and partial differential equations. It is explicitly shown in [17] that the following definition is for ordinary differential equations equivalent to the one given above.

Definition 13. The *involution index* ν_i of a system of differential equations is the minimal number of steps required by the Cartan–Kuranishi completion to an involutive system.

But we saw above that for estimating the perturbation index the determinacy index $\tilde{\nu}_d$ and not the involution index $\tilde{\nu}_i$ is decisive. Thus the question arises how this index can be generalised. [17] gave only a purely formal answer to this. In our discussion above it has clearly emerged that the fundamental concept behind the determinacy index is the underlying equation. We will demonstrate how this notion can be extended to partial differential equations using the Cartan normal form of an involutive system.

Our claim is that in the Cartan normal form the subsystem consisting of all equations of class n may be considered as an underlying equation. We will restrict in the sequel to systems which are not underdetermined, i.e. where $\beta^{(n)} = m$. This is only for simplifying the presentation and not because of any fundamental problems with underdetermined systems. Under this assumption, the subsystem of class n is normal, i.e. in Cauchy–Kovalevskaya form. An important property of an underlying equation of a differential algebraic equation is that the constraint manifold is invariant under its flow. We show now that our subsystem satisfies a similar property; however, for a rigorous proof we restrict to linear systems with analytic coefficients.

Proposition 14. *Let (11) be an involutive system and let the functions $\phi_{k\alpha}$ be linear in $\boldsymbol{u}$, $\partial\boldsymbol{u}/\partial\boldsymbol{x}$ and analytic in $\boldsymbol{x}$. If the hyperplane $x_n = 0$ is non-characteristic and $\boldsymbol{U}(\boldsymbol{x})$ is a smooth solution of the subsystem of class n which satisfies for $x_n = 0$ all other equations in the system, then it satisfies these equations for all values of x_n.*

Proof. We must analyse what happens upon entering the solution $\boldsymbol{U}(\boldsymbol{x})$ into the equations of lower class. So we introduce for $1 \leq k < n$ and $1 \leq \alpha \leq \beta^{(k)}$ the residuals

$$\Delta_{k\alpha}(\boldsymbol{x}) = \frac{\partial U_\alpha}{\partial x_k}(\boldsymbol{x}) - \phi_{k\alpha}\left(\boldsymbol{x}, \boldsymbol{U}(\boldsymbol{x}), \frac{\partial \boldsymbol{U}}{\partial \boldsymbol{x}}(\boldsymbol{x})\right) . \tag{15}$$

Obviously, they are smooth functions. The proposition claims that if the functions $\Delta_{k\alpha}$ vanish for $x_n = 0$, they vanish for all values of x_n. We will prove it by deriving a system of partial differential equations for the residuals.

By assumption, (11) is a linear involutive system with analytic coefficients. As x_n is a non-multiplicative variable for all equations of class less than n, there must exist analytic functions $A_{i\beta j}(\boldsymbol{x})$ and $B_{i\beta}(\boldsymbol{x})$ such that[4]

$$\begin{aligned}\frac{\partial}{\partial x_n}\left(\frac{\partial u_\alpha}{\partial x_k} - \phi_{k\alpha}(\boldsymbol{x},\boldsymbol{u},\partial\boldsymbol{u}/\partial\boldsymbol{x})\right) = \\ \sum_{i=1}^{n}\sum_{\beta=1}^{\beta^{(i)}}\sum_{j=1}^{i} A_{i\beta j}(\boldsymbol{x})\frac{\partial}{\partial x_j}\left(\frac{\partial u_\beta}{\partial x_i} - \phi_{i\beta}(\boldsymbol{x},\boldsymbol{u},\partial\boldsymbol{u}/\partial\boldsymbol{x})\right) + \\ \sum_{i=1}^{n}\sum_{\beta=1}^{\beta^{(i)}} B_{i\beta}(\boldsymbol{x})\left(\frac{\partial u_\beta}{\partial x_i} - \phi_{i\beta}(\boldsymbol{x},\boldsymbol{u},\partial\boldsymbol{u}/\partial\boldsymbol{x})\right).\end{aligned} \tag{16}$$

Entering our solution $\boldsymbol{U}(\boldsymbol{x})$ into these relations yields the following *linear* homogeneous system with analytic coefficients for the residuals $\Delta_{k\alpha}$

$$\frac{\partial\Delta_{k\alpha}}{\partial x_n} = \sum_{i=1}^{n-1}\sum_{\beta=1}^{\beta^{(i)}}\left\{\sum_{j=1}^{i} A_{i\beta j}(\boldsymbol{x})\frac{\partial\Delta_{i\beta}}{\partial x_j} + B_{i\beta}(\boldsymbol{x})\Delta_{i\beta}\right\}. \tag{17}$$

Here the summands for $i = n$ disappeared, as the function $\boldsymbol{U}(\boldsymbol{x})$ is a solution of the subsystem of class n. The system (17) is in Cauchy–Kovalevskaya form and we are interested in its smooth solutions for the initial conditions $\Delta_{k\alpha}(x_n = 0) = 0$, as by assumption $\boldsymbol{U}$ satisfies for $x_n = 0$ all equations of lower class. Because of the homogeneity, $\Delta_{k\alpha}(\boldsymbol{x}) = 0$ is obviously a solution. As (17) is analytic, the Holmgren Theorem tells us that this is the only smooth solution of the initial value problem. Thus the residuals vanish for all values of x_n and $\boldsymbol{U}(\boldsymbol{x})$ is a smooth solution of the full system (11). □

For simplicity, we have formulated this proposition only for smooth solutions. In fact, the Holmgren Theorem gives uniqueness for $\mathcal{C}^1$ solutions (there are also extensions to Sobolev spaces). So the result holds for much larger classes of solutions including weak ones. Unfortunately, the restriction to analytic coefficients is not so easy to remove.[5] The Calderon Theorem provides uniqueness also for linear systems with smooth coefficients; however, the system must satisfy some technical conditions [21,23]. In any case, the above used technique of proof reduces the problem even for *non-linear* systems always to the question of the uniqueness of the zero solution of a *linear* homogeneous system.

[4] For general systems, the functions $\boldsymbol{A}$ and $\boldsymbol{B}$ could also depend on $\boldsymbol{u}$ and $\partial\boldsymbol{u}/\partial\boldsymbol{x}$. Because of the assumed linearity of (11) it suffices to consider here functions depending only on $\boldsymbol{x}$.

[5] This is also the main obstacle for extending Proposition 14 to non-linear systems. Only for analytic solutions of analytic systems the extension is straightforward using the Cauchy–Kovalevskaya Theorem.

Definition 15. The *determinacy index* ν_d of a system of differential equations is the number of steps required in the Cartan–Kuranishi completion until a system is reached with $\beta^{(n)} = m$.

As in the case of differential algebraic equations, the determinacy index is only defined for systems which are not underdetermined, as otherwise the condition $\beta^{(n)} = m$ will never be satisfied.[6] The idea behind this definition is exactly the same as in Definition 4: as soon as the condition is met, we know an underlying equation. Under additional hypotheses one can now derive results similar to Theorem 10. We give here only a simple example.

Theorem 16. *Let (11) be as in Proposition 14 and let again the hyperplane* $x_n = 0$ *be non-characteristic for it. Assume furthermore that the right hand side of the subsystem of class* n *can be written in the form* $A\boldsymbol{u}$ *where* A *is the generator of a strongly continuous semigroup acting on an appropriately chosen Banach space. Then* $\tilde{\nu}_d \leq \nu_p \leq \tilde{\nu}_d + 1$.

Proof. We slightly change our notation: we write t for x_n and $\dot{\boldsymbol{u}}$ for the partial derivatives $\partial\boldsymbol{u}/\partial x_n$. Then we may rewrite the subsystem of class n as $\dot{\boldsymbol{u}} = A\boldsymbol{u} + \boldsymbol{\delta}$ where $\boldsymbol{u}$ and $\boldsymbol{\delta}$ are considered as functions of t taking values in the chosen Banach space. It is an elementary result in the theory of strongly continuous semigroups [10,14] that the solution $\hat{\boldsymbol{u}}(t)$ of this equation is

$$\hat{\boldsymbol{u}}(t) = S(t)\hat{\boldsymbol{u}}(0) + \int_0^t S(t-\tau)\boldsymbol{\delta}(\tau)d\tau \tag{18}$$

where $S(t)$ is the semigroup generated by A and $\hat{\boldsymbol{u}}(0)$ an arbitrary element of the domain of A.

Now we can easily prove an analogous estimate to Proposition 9. We compare the solution $\hat{\boldsymbol{u}}(t)$ given by (18) with a solution $\boldsymbol{u}(t)$ for $\boldsymbol{\delta}(t) = 0$. If $\|\cdot\|$ denotes the norm of the used Banach space, we get

$$\|\boldsymbol{u}(t) - \hat{\boldsymbol{u}}(t)\| \leq C\left(\|\boldsymbol{u}(0) - \hat{\boldsymbol{u}}(0)\| + \int_0^t \|\boldsymbol{\delta}(\tau)\|\, d\tau\right) \tag{19}$$

for some constant C, as the norms of the operators $S(t)$ of a strongly continuous semigroup are bounded on any finite interval.

As the original system has the perturbed determinacy index $\tilde{\nu}_d$, the perturbations $\boldsymbol{\delta}$ in (18) depend on derivatives of this order of the perturbations of

[6] One can easily verify that if $\beta^{(n)} < m$ in an involutive system, then some of the components of $\boldsymbol{u}$ can be chosen completely arbitrary, and thus this condition represents a simple criterion for an underdetermined system of partial differential equations [15]. The classical "counting rule" comparing the number of equations with the number of unknowns can be misleading, as the Yang-Mills equations show: although they comprise as many equations as unknowns, they are underdetermined due to the gauge symmetry.

the original system. Thus we have $\nu_p \leq \tilde{\nu}_d + 1$. By exactly the same reasoning as in the finite-dimensional case it can happen, if $S(t)$ possess a particularly simple structure, that the integration can be performed within the norm and thus kills one differentiation. Hence $\nu_p \geq \tilde{\nu}_d$. □

7 Conclusions

We have shown that the formal theory of differential equations provides a convenient framework for analysing very general systems. In particular, as it is a theory for partial differential equations it allows us to present classical results on differential algebraic equations in a manner that can be straightforwardly extended to multivariate systems.

Of course, it is much harder to derive estimates of solutions of partial differential equations than of solutions of ordinary differential equations. Thus one cannot expect to obtain results in the same generality as Theorem 10. However, using the formal theory we have at least reduced the problem to analysing a normal system instead of an overdetermined one. A huge amount of results is available on such systems. We applied here only some elementary semigroup theory, as it permits us to treat the given partial differential equation as an ordinary differential equation on an appropriately chosen Banach space and thus to straightforwardly extend Theorem 10.

The key for extending our approach to larger classes of systems is Proposition 14. Its proof depends on the uniqueness of the zero solution of (17). Thus the crucial point is to understand the properties of this system which may be considered as a differential form of syzygies. An example of how it inherits properties of the original system can be found in [20]. There systems are studied where the underlying equation is (strictly) hyperbolic and the constraints form an elliptic system. Then one can show that (17) is again a (strictly) hyperbolic system.

Most approaches to indices for general systems of partial differential equations are based on first reducing the system to a differential algebraic equation by semi-discretisations, integral transforms or something similar and then determining an index of this differential algebraic equation. In [20] we have shown for linear systems with constant coefficients how the thus obtained indices are related to indices directly determined from the original system of partial differential equations.

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