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Involutive Bases in the Weyl Algebra*

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Abstract

We discuss the definition and the construction of involutive bases, a special kind of Gröbner bases, for left ideals in the Weyl algebra. We consider not only term orders, for which the extension is straightforward, but also the more general case of multiplicative monomial orders.

1. Introduction

Involutive bases are a special kind of non-reduced Gröbner bases. They have been introduced for polynomial ideals by Gerdt and collaborators (see e.g. [Zharkov and Blinkov, 1993, Gerdt and Blinkov, 1998a,b]) based on ideas from the Janet-Riquier theory of differential equations [Janet, 1929, Riquier, 1910]. Involutive bases possess special combinatorial properties: in particular, they define Stanley decompositions [Seiler, 2001]. These decompositions were originally introduced for Hilbert function computations [Stanley, 1978], but also proved useful in other applications like invariant theory [Sturmfels and White, 1991, Gatermann, 2000] or the computation of syzygy resolutions [Seiler, 2001].

The Janet-Riquier theory also motivated an explicit algorithm for the determination of involutive bases. Experiments by Gerdt et al. [2001] showed that it is fairly efficient and represents a highly competitive alternative to the traditional Buchberger algorithm for the computation of Gröbner bases, even if one is not interested in the additional combinatorial properties of involutive bases.

Thus it appears natural to see whether involutive bases can also be introduced for ideals in other than commutative polynomial rings. This article discusses

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the case of left ideals in the *Weyl algebra*† [Coutinho, 1995]. The algorithmic study of $\mathcal{D}$-modules has been pioneered by Briançon and Maisonobe [1984] in the univariate and by Castro-Jiménez [1984, 1987] and Galligo [1985] in the multivariate case. Further references can be found in [Saito et al., 2000].

For term orders such a generalisation is rather straightforward. It was remarked by Apel [1995, 1998] that the theory can be extended to algebras of solvable type which include in particular Ore algebras. A more general class of solvable algebras was discussed by Seiler [2001]; the special case of linear differential operators was extensively studied by Gerdt [1999].

The situation for the more general multiplicative monomial orders is considerably more difficult, as for them normal form algorithms do not necessarily terminate. One particularly important kind of multiplicative monomial orders which are not term orders is specified by means of weight vectors of the form $(-\xi, \xi) \in \mathbb{R}^{2n}$. They arise from the action of the algebraic torus $(\mathbb{K}^*)^n$ on the Weyl algebra and are, for example, considered in connection with b-functions (Bernstein-Sato polynomial) whose roots are of interest for various algorithms dealing with $\mathcal{D}$-modules (cf. [Saito et al., 2000, Section 5] or [Oaku et al., 2000]). Furthermore, the classical Fuchsian conditions for regular singular points [Ince, 1956] may be expressed via the initial ideals with respect to such non-term orders and generalised from scalar linear ordinary differential equations to linear systems of partial differential equations (see e. g. Assi et al. [1996]).

For Gröbner bases, the problem of non-terminating normal form algorithms is circumvented by computing in the homogenised Weyl algebra: any multiplicative monomial order on the Weyl algebra can be lifted to a term order on the homogenised Weyl algebra. For commutative polynomials, this trick goes back at least to Lazard [1983]; in the Weyl algebra it has been used for example by Castro-Jiménez and Narváez-Macarro [1997]. As involutive bases do not only depend on an order but also on an involutive division, we are faced with the additional problem of lifting this division to the homogenised Weyl algebra. The main point of this article is to show that this is indeed possible.

In the context of non-term orders, the distinction between weak and strong involutive bases becomes important. Usually, one is only interested in the latter ones, as only they lead to direct sum decompositions and consequently to all the interesting properties of involutive bases. However, like reduced Gröbner bases, strong involutive bases require some normal form computations which may not be possible for a non-term order. Nevertheless, we are able to prove the existence of weak involutive bases which are simultaneously Gröbner bases.

The article is organised as follows. The next section introduces the notion of an involutive division. For easier applicability in different algebras, we will not do this using terms but multi indices. Sect. 3 collects some basic facts about the Weyl algebra and the homogenised Weyl algebra. Sect. 4 contains the definition

†It should be noted that within the theory of $\mathcal{D}$-modules another notion of involutivity exists [Oaku, 1997] which is not related to ours.

of involutive bases for the Weyl algebra and proves some of their basic properties in the case of a term order. The following two sections discuss the extension of these results to non-term orders: first the lift of an involutive division on $\mathbb{N}_0^{2n}$ to a division on $\mathbb{N}_0^{2n+1}$ is introduced; then (weak) involutive bases and Gröbner bases are constructed via a homogenisation. Finally, some conclusions are given.

2. Involutive Divisions on $\mathbb{N}_0^n$

The concept of an involutive division was introduced for (multivariate) polynomials over a field $\mathbb{K}$, i.e. in the ring $\mathbb{K}[x_1, \dots, x_n]$, and leads to a restriction of the divisibility relation on power products. In order to allow for an easy transfer to other domains, we point out the underlying ideas on the set of n-tuples with non-negative integer entries. At the beginning this may lead to a slight confusion: while sticking to the term "division", we now use addition and subtraction.

We call an element of the Abelian monoid $(\mathbb{N}_0^n, +)$ a *multi index*. For two multi indices $\mu = (\mu^1, \dots, \mu^n)$ and $\nu = (\nu^1, \dots, \nu^n) \in \mathbb{N}_0^n$, we define $\mu|\nu$ (μ *divides* ν), if $\mu^i \leq \nu^i$ for all i. The *cone* of μ with respect to a subset $N \subseteq \{1, \dots, n\}$ is the set $C_N(\mu) = \mu + \mathbb{N}_N^n \subset \mathbb{N}_0^n$ where $\mathbb{N}_N^n = \{\nu \in \mathbb{N}_0^n : \nu^i = 0 \text{ for } i \notin N\}$.

Definition: An *involutive division* L on $\mathbb{N}_0^n$ is given by prescribing for each finite subset $\mathcal{N} \subset \mathbb{N}_0^n$ and for each multi index $\mu \in \mathcal{N}$ a set $N_{L,\mathcal{N}}(\mu)$ of *multiplicative indices* such that the following holds:

- If $C_{L,\mathcal{N}}(\mu)$ is used as a shorthand for $C_{N_{L,\mathcal{N}}(\mu)}(\mu)$, then for all $\mu, \nu \in \mathcal{N}$ with $C_{L,\mathcal{N}}(\mu) \cap C_{L,\mathcal{N}}(\nu) \neq \emptyset$ either $C_{L,\mathcal{N}}(\mu) \subseteq C_{L,\mathcal{N}}(\nu)$ or $C_{L,\mathcal{N}}(\nu) \subseteq C_{L,\mathcal{N}}(\mu)$.
- If $\mathcal{M} \subset \mathcal{N}$, then $\forall \mu \in \mathcal{M} : N_{L,\mathcal{N}}(\mu) \subseteq N_{L,\mathcal{M}}(\mu)$.

$C_{L,\mathcal{N}}(\mu)$ is called the *involutive cone* of μ with respect to L and $\mathcal{N}$. We denote the complement of $N_{L,\mathcal{N}}(\mu)$ in $\{1, \dots, n\}$, the *non-multiplicative indices* of μ, by $\bar{N}_{L,\mathcal{N}}(\mu)$. Finally, for $\mu \in \mathcal{N}$ and $\nu \in \mathbb{N}_0^n$ we write $\mu \,|_{L,\mathcal{N}}\, \nu$ (μ *involutively divides* or is an *involutive divisor* of ν), if and only if $\nu \in C_{L,\mathcal{N}}(\mu)$.

We include at this point the definitions of the two involutive divisions most frequently encountered. The above conditions are easily verified for them. It suffices to state how the multiplicative indices are determined.

- *Janet division* J: Let $\mathcal{N}$ be a finite subset of $\mathbb{N}_0^n$ and $\mu \in \mathcal{N}$.
  - $1 \in N_{J,\mathcal{N}}(\mu)$, if and only if $\mu^1 \geq \nu^1$ for all $\nu \in \mathcal{N}$.
  - For $i = 2, \dots, n$ we have $i \in N_{J,\mathcal{N}}(\mu)$, if and only if $\mu^i \geq \nu^i$ for all those $\nu \in \mathcal{N}$ with $\nu^\ell = \mu^\ell$ for $\ell = 1, \dots, i-1$.
- *Pommaret division* P: Let $\mu \in \mathbb{N}_0^n$ be an arbitrary multi index and let k be the position of the right-most non-zero entry. Then i is a multiplicative index, if and only if $i \geq k$. For $\mu = (0, \dots, 0)$, all indices are multiplicative.

In the Pommaret division the multiplicative indices are determined independently of a set $\mathcal{N}$. Such divisions are called *globally defined.* Note that for both mentioned divisions it is important that multi indices are ordered tuples, as the entries are considered one after the other. Thus we may introduce variants of them by applying a fixed permutation to any multi index before computing its multiplicative indices. In the context of partial differential equations it is for example customary to revert the ordering of the entries, i. e. to apply the permutation $(n\ n-1\ \cdots\ 1)$.

Definition: Let L be an involutive division on $\mathbb{N}_0^n$ and $\mathcal{N} \subset \mathbb{N}_0^n$ a finite set.

1. The *involutive span* of $\mathcal{N}$ is the union of the involutive cones of its elements:
$$\langle\, \mathcal{N}\,\rangle_L = \bigcup_{\mu\in\mathcal{N}} C_{L,\mathcal{N}}(\mu)\,. \tag{1}$$
2. The set $\mathcal{N}$ is *involutively autoreduced* (with respect to the division L), if $C_{L,\mathcal{N}}(\mu) \cap C_{L,\mathcal{N}}(\nu) = \emptyset$ for all $\mu \neq \nu \in \mathcal{N}$.
3. $\mathcal{N}$ is called *weakly (L)-involutive*, if $\langle\, \mathcal{N}\,\rangle_L = \langle\, \mathcal{N}\,\rangle$ where $\langle\, \mathcal{N}\,\rangle$ denotes the ordinary span of $\mathcal{N}$, i. e. the monoid ideal $\langle\, \mathcal{N}\,\rangle = \mathcal{N} + \mathbb{N}_0^n$.
4. A finite subset $\hat{\mathcal{N}} \subset \langle\, \mathcal{N}\,\rangle$ is called a *weak involutive basis* of $\langle\, \mathcal{N}\,\rangle$, if $\langle\, \hat{\mathcal{N}}\,\rangle_L = \langle\, \mathcal{N}\,\rangle$. If $\hat{\mathcal{N}}$ contains $\mathcal{N}$, it is a *weak (L)-completion* of $\mathcal{N}$. If the set $\hat{\mathcal{N}}$ is furthermore autoreduced, it is a *(strong) involutive basis* of $\langle\, \mathcal{N}\,\rangle$ or a *(strong) completion* of $\mathcal{N}$, respectively.
5. An involutive division L is called *Noetherian*, if every finite set $\mathcal{N} \subset \mathbb{N}_0^n$ of multi indices has a completion.

For an involutively autoreduced set the union in (1) is disjoint. While the Janet division is Noetherian, the Pommaret division is not.

LEMMA 2.1: *If $\mathcal{N}$ is a weakly involutive set, there exists a subset $\mathcal{N}' \subset \mathcal{N}$ such that $\mathcal{N}'$ is a strong involutive basis of $\langle\, \mathcal{N}\,\rangle$.*

Proof: The proof represents a nice motivation for the two conditions in the definition of an involutive division. If $\mathcal{N}$ is not yet a strong involutive basis, the union in (1) cannot be disjoint. Thus there must exist involutive cones which intersect. By the first condition, this implies that some cones are contained in other ones; no other form of intersection is possible. If we eliminate the corresponding elements of $\mathcal{N}$, we get a subset $\mathcal{N}' \subset \mathcal{N}$ which has by the second condition the same involutive span, as the remaining elements may only gain additional multiplicative indices. Thus after a finite number of such eliminations we get a strong involutive basis. □

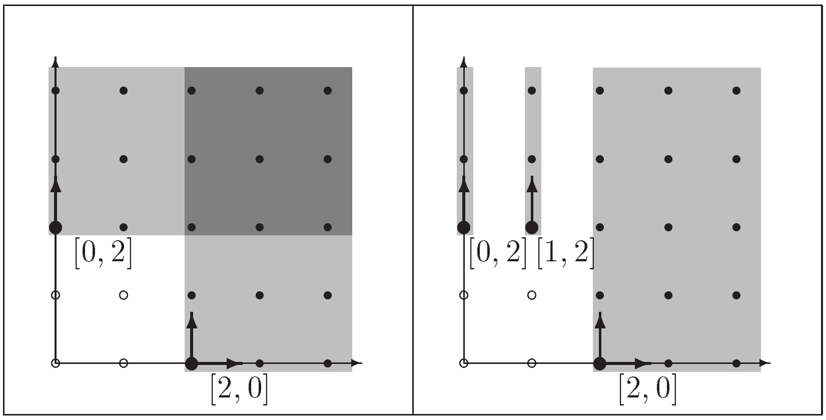

Figure 1: *Left:* intersecting cones. *Right:* involutive cones.

Fig. 1 demonstrates the geometric interpretation of involutive divisions for $n = 2$. In the left diagram one can see the span of the set $\mathcal{N} = \{[0, 2], [2, 0]\}$; the vertices belonging to it are marked by black points. Obviously, the full cones of the two elements of $\mathcal{N}$ intersect in the darkly shaded area. The arrows represent the multiplicative indices, i.e. the “allowed directions”, for both the Janet and the Pommaret division (they are identical for this example). The set $\mathcal{N}$ is not (weakly) involutive, as for every $k \geq 2$ the multi index $[1, k]$ belongs to $\langle \mathcal{N} \rangle$ but not to the involutive span. The right diagram shows an involutive basis of the monoid ideal $\langle \mathcal{N} \rangle$ for the Janet and the Pommaret division. We must add to $\mathcal{N}$ the multi index $[1, 2]$ and both for this element and for $[0, 2]$ only the index 2 is multiplicative. One can clearly see how $\langle \mathcal{N} \rangle$ is decomposed into three disjoint cones: one of dimension 2, two of dimension 1.

In order to obtain an efficient tool for finding involutive completions of a given set, one requires the involutive division L to possess two additional and rather technical properties introduced by Gerdt and Blinkov [1998a].

Definition: Let L be an involutive division on $\mathbb{N}_0^n$.

- L is *continuous*, if for all finite subsets $\mathcal{N} \subset \mathbb{N}_0^n$ the following condition is satisfied: for each finite sequence $(\mu_i)_{i=1,\dots,r}$ of elements from $\mathcal{N}$ obeying that for all i the sum‡ $\mu_i + 1_j \in C_{L,\mathcal{N}}(\mu_{i+1})$ for some non-multiplicative index $j \in \bar{N}_{L,\mathcal{N}}(\mu_i)$, the inequality $\mu_k \neq \mu_\ell$ holds for all $k \neq \ell$.
- L is *constructive*, if it satisfies the following condition: for all finite subsets $\mathcal{N} \subset \mathbb{N}_0^n$ and for each multi index $\mu \in \mathcal{N}$ and $i \in \bar{N}_{L,\mathcal{N}}(\mu)$ with
  1. $\mu + 1_i \notin \langle \mathcal{N} \rangle_L$,
  2. $\mu + 1_i$ is minimal in the sense that each of its proper divisors lies in the involutive span of $\mathcal{N}$, i.e. if there exists a $\nu \in \mathcal{N}$ and a $j \in \bar{N}_{L,\mathcal{N}}(\nu)$ such that $\nu + 1_j \mid \mu + 1_i$ and $\nu + 1_j \neq \mu + 1_i$, then $\nu + 1_j \in \langle \mathcal{N} \rangle_L$,

‡The multi index 1_i has 1 in the i^{th} place and all other entries are 0.

$\mathcal{N}$ cannot be enlarged by a multiplicative multiple to include $\mu + 1_i$, i. e. there does not exist a $\rho \in \langle \mathcal{N} \rangle_L$ such that $\mu + 1_i \in \langle \mathcal{N} \cup \{\rho\} \rangle_L$.

It was shown by Gerdt and Blinkov [1998a] that for a continuous division L a set $\mathcal{N} \subset \mathbb{N}_0^n$ is weakly L-involutive, if it is *locally L-involutive*, i. e. if for every $\mu \in \mathcal{N}$ and $i \in \bar{N}_{L,\mathcal{N}}(\mu)$, there exists some $\nu \in \mathcal{N}$ with $\mu + 1_i \in C_{L,\mathcal{N}}(\nu)$. This observation leads to a simple strategy for completing a set $\mathcal{N}$ [Gerdt and Blinkov, 1998a, Calmet et al., 2001]. If $i \in \bar{N}_{L,\mathcal{N}}(\mu)$ is a non-multiplicative index for $\mu \in \mathcal{N}$, we check whether $\mu + 1_i \in \langle \mathcal{N} \rangle_L$. If not, $\mu + 1_i$ is added to $\mathcal{N}$.

More precisely, we obtain the algorithm shown in Fig. 2. The `repeat` loop determines a weak involutive basis. In the last step we perform an involutive autoreduction based on the proof of Lemma 2.1 and obtain a strong basis. An extended version of it will appear later, when we compute involutive bases in the Weyl algebra.

Algorithm: *Involutive completion in* $\mathbb{N}_0^n$
Input: Finite subset $\mathcal{N} \subset \mathbb{N}_0^n$, involutive division L, term order $\preceq$ on $\mathbb{N}_0^n$
Output: Involutive basis of $\langle \mathcal{N} \rangle$

`repeat`
  $\mathcal{S} \leftarrow \{\mu + 1_i \;:\; \mu \in \mathcal{N},\, i \in \bar{N}_{L,\mathcal{N}}(\mu),\, \mu + 1_i \notin \langle \mathcal{N} \rangle_L\}$
  $\mathcal{N} \leftarrow \mathcal{N} \cup \{\min_{\preceq} \mathcal{S}\}$
`until` $\mathcal{S} = \emptyset$
`return InvAutoReduce`$_L(\mathcal{N})$

Figure 2: Completion in $\mathbb{N}_0^n$

Proposition 2.1: [Gerdt and Blinkov, 1998a] *If the division L is continuous and constructive, the algorithm shown in Fig. 2 will terminate with an L-completion for any finite input set $\mathcal{N}$ possessing an L-completion.*

One can show that the output of this algorithm is independent of the chosen term order $\preceq$ (in fact, one may even choose in each iteration of the `repeat` loop a different order). The term order only serves to realise a selection strategy corresponding to the well-known normal selection strategy in Buchberger's algorithm. The use of this strategy is crucial for the proof of Prop. 2.1.

3. The Weyl Algebra and Its Homogenisation

We recall some facts about Gröbner bases in the Weyl algebra and in its homogenisation; for more details we refer to the book of Saito et al. [2000]. By $\mathbb{K}$ we denote a field of characteristic zero; for computational purposes we assume that all occurring computations in $\mathbb{K}$ can be performed effectively.

Definition: For $n \in \mathbb{N}$ the *n-dimensional Weyl algebra* $\mathbb{W}_n$ is the free associative $\mathbb{K}$-algebra with generators $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ modulo the relations

$$x_i x_j = x_j x_i\,, \quad \partial_i \partial_j = \partial_j \partial_i\,, \quad \partial_i x_j = x_j \partial_i + \delta_{ij}\,, \qquad (1 \leq i, j \leq n)$$

where, as usual, $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. Adding another generator h that commutes with $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ and replacing δ_{ij} with $\delta_{ij} h^2$ in the above relations yields the *n-dimensional homogenised Weyl algebra* $\mathbb{W}_n^{(h)}$.

Any element $f \in \mathbb{W}_n$ can be written in a unique *normally ordered form*

$$f = \sum\nolimits_{\mu,\nu \in \mathbb{N}_0^n} a_{\mu\nu} x^\mu \partial^\nu \tag{2}$$

where only finitely many $a_{\mu\nu} \in \mathbb{K}$ do not vanish. So the normally ordered *terms* $\mathbb{T}_n = \{x^\mu \partial^\nu = x_1^{\mu^1} \cdots x_n^{\mu^n} \partial_1^{\nu^1} \cdots \partial_n^{\nu^n} : \mu, \nu \in \mathbb{N}_0^n\}$ form a $\mathbb{K}$-vector space basis of $\mathbb{W}_n$. Similarly, $\mathbb{T}_n^{(h)} = \{h^\lambda x^\mu \partial^\nu : \lambda \in \mathbb{N}_0,\, \mu, \nu \in \mathbb{N}_0^n\}$ forms a basis of $\mathbb{W}_n^{(h)}$.

For both Gröbner bases and the involutive bases defined below, the concepts of multiplicative monomial orders and term orders, respectively, are essential.

Definition: Let $\preceq$ be a total order on $\mathbb{T}_n$. We call $\preceq$ a *multiplicative monomial order* (on $\mathbb{T}_n$), if

1. $1 \preceq x_i \partial_i$ for all $1 \leq i \leq n$ and
2. $x^\alpha \partial^\beta \preceq x^a \partial^b$ implies $x^{\alpha+\mu} \partial^{\beta+\nu} \preceq x^{a+\mu} \partial^{b+\nu}$ for all $\mu, \nu \in \mathbb{N}_0^n$.

If in addition $1 \preceq t$ holds for all $t \in \mathbb{T}_n$, we call $\preceq$ a *term order* (on $\mathbb{T}_n$); otherwise we refer to $\preceq$ as a *non-term order.*

Replacing the first of the above conditions with $h^2 \preceq x_i \partial_i$ and adapting the second one in the obvious way, we obtain the corresponding definitions for the terms $\mathbb{T}_n^{(h)}$ in the homogenised Weyl algebra.

Example: Let $(\xi, \zeta) \in \mathbb{R}^{2n}$ be a weight vector, i. e. $\xi + \zeta \in \mathbb{R}^n$ is non-negative, and $\preceq$ an arbitrary term order. We define a multiplicative monomial order $\preceq_{(\xi,\zeta)}$ by setting $x^\alpha \partial^\beta \prec_{(\xi,\zeta)} x^a \partial^b$, if either $\alpha \cdot \xi + \beta \cdot \zeta < a \cdot \xi + b \cdot \zeta$ or $\alpha \cdot \xi + \beta \cdot \zeta = a \cdot \xi + b \cdot \zeta$ and $x^\alpha \partial^\beta \prec x^a \partial^b$. This yields a term order, if and only if (ξ, ζ) is non-negative.

After fixing a multiplicative monomial order $\preceq$ on $\mathbb{T}_n$, we define the *leading term*, *leading exponent*, and *leading coefficient* of $f \in \mathbb{W}_n \setminus \{0\}$ by selecting in the normally ordered form (2) the $\preceq$-maximal term occurring in f. This yields maps $\mathrm{lt}_\preceq : \mathbb{W}_n \setminus \{0\} \longrightarrow \mathbb{T}_n$, $\mathrm{le}_\preceq : \mathbb{W}_n \setminus \{0\} \longrightarrow \mathbb{N}_0^{2n}$, and $\mathrm{lc}_\preceq : \mathbb{W}_n \setminus \{0\} \longrightarrow \mathbb{K}$ (we omit the subscript $\preceq$ if there is no doubt about the order used). Analogously, for $\mathbb{W}_n^{(h)}$ we can define maps $\mathrm{lt}_\preceq^{(h)}$, $\mathrm{le}_\preceq^{(h)}$, and $\mathrm{lc}_\preceq^{(h)}$. Finally, we introduce maps $\epsilon : \mathbb{T}_n \longrightarrow \mathbb{N}_0^{2n}$ and $\iota : \mathbb{N}_0^{2n} \longrightarrow \mathbb{T}_n$ to switch between a term and the corresponding multi index.

A crucial prerequisite for later defining involutive bases in $\mathbb{W}_n$ is the following result whose proof is a "simple computation" and hence omitted here.

PROPOSITION 3.1: *For $f, g \in \mathbb{W}_n$ we have* $\mathrm{le}(f \cdot g) = \mathrm{le}(f) + \mathrm{le}(g)$.

As for the commutative polynomial ring, one introduces the notion of a Gröbner basis of a left ideal in $\mathbb{W}_n$ with respect to a term order. By means of suitable analogues of the commutative S(yzygy)-polynomials and the commutative normal form algorithm, such a Gröbner basis can be computed with Buchberger's algorithm. Furthermore, it is well-known that in the Weyl algebra the notion of a Gröbner basis can be generalised to multiplicative monomial orders.

Definition: Let I be a left ideal in $\mathbb{W}_n$ and $\preceq$ a multiplicative monomial order on $\mathbb{T}_n$. A finite subset $G \subset \mathbb{W}_n$ with $0 \notin G$ is a *Gröbner basis* of I, if

- $\mathbb{W}_n \cdot G = I$ and
- $\langle \{\mathrm{le}(g) : g \in G\} \rangle = \{\mathrm{le}(f) : f \in I\} = \mathrm{le}\,(I)$.[§]

A Gröbner basis G is *reduced*, if for any two distinct $f, g \in G$ the following condition holds: for all terms $t \in \mathbb{T}_n$ occurring in the normally ordered form (2) of f with non-zero coefficient, we have $\mathrm{le}(g) \nmid \epsilon(t)$.

Saito et al. [2000] show how to compute effectively Gröbner bases in $\mathbb{W}_n$ with respect to non-term orders $\preceq$; the key idea is to derive from $\preceq$ a term order $\preceq_h$ on $\mathbb{T}_n^{(h)}$ that respects the total degree:

$$h^\lambda x^\alpha \partial^\beta \preceq_h h^\ell x^a \partial^b \iff \begin{array}{l} \lambda + |\alpha| + |\beta| < \ell + |a| + |b| \text{ or both} \\ \lambda + |\alpha| + |\beta| = \ell + |a| + |b| \text{ and } x^\alpha \partial^\beta \preceq x^a \partial^b \end{array}$$

(with $|\mu| = \sum_{i=1}^n \mu^i$ for a multi index $\mu = (\mu^1, \ldots, \mu^n) \in \mathbb{N}_0^n$).

Using this term order we can derive the required Gröbner basis by applying Buchberger's algorithm to the homogenisation of any given finite generating set of the ideal under consideration. Here the homogenisation $f^{(h)} \in \mathbb{W}_n^{(h)}$ of $f \in \mathbb{W}_n$ is obtained as usual by inserting suitable powers of h in the normally ordered form of f. Using the obvious definition for Gröbner bases in $\mathbb{W}_n^{(h)}$, we have the following theorem (see [Saito et al., 2000, Theorem 1.2.4]):

[§]This condition can equivalently be expressed as an equality of monomial ideals in a commutative polynomial ring (cf. [Saito et al., 2000]). However, having in mind involutive bases, we prefer the formulation with multi indices.

THEOREM 3.1: *Let $F \subset \mathbb{W}_n$ be a finite generating set of some left ideal $I \subseteq \mathbb{W}_n$, $\preceq$ a multiplicative monomial order on $\mathbb{T}_n$, and $F^{(h)} \subset \mathbb{W}_n^{(h)}$ the set obtained by homogenising the elements in F. Then applying Buchberger's algorithm to $F^{(h)}$ and the induced term order $\preceq_h$ on $\mathbb{T}_n^{(h)}$ yields a set $\tilde{G} \subset \mathbb{W}_n^{(h)}$ whose dehomogenisation G is a Gröbner basis of I with respect to $\preceq$.*

Remark: The proof of Saito et al. [2000] shows that it is irrelevant how to obtain the Gröbner basis $\tilde{G}$. Any Gröbner basis of the left ideal $\langle F^{(h)} \rangle$ with respect to $\preceq_h$ yields by dehomogenisation a Gröbner basis of I with respect to $\preceq$.

4. Involutive Bases

We turn to the problem of introducing involutive bases for left ideals in the Weyl algebra. Our starting point is the choice of some order $\preceq$ on $\mathbb{T}_n$, either a term order or a multiplicative monomial order, for selecting the leading term of an element. In the latter case some computations might not be possible, as the termination of normal form algorithms cannot be ensured. We will treat this problem in the next two sections; for the following definitions it does not matter.

Definition: Let $F = \{f_1, \dots, f_r\} \subset \mathbb{W}_n$ be a finite set. With respect to a given involutive division L on $\mathbb{N}_0^{2n}$, the *involutive span* of F is defined as

$$\langle F \rangle_L = \sum_{i=1}^{r} \mathbb{K}[(\iota \circ N_{L,\mathrm{le}(F)} \circ \mathrm{le})(f_i)] \cdot f_i \,. \tag{3}$$

By $\langle F \rangle$ we denote the left ideal $\mathbb{W}_n \cdot F$ in $\mathbb{W}_n$. The set F is a *weak involutive basis* of the left ideal $I \subseteq \mathbb{W}_n$, if $\langle F \rangle_L = \langle F \rangle = I$.

Details on the notions of reducibility and normal form in the Weyl algebra can be found in [Saito et al., 2000]. Note that Prop. 3.1 allows us the simple generalisation of these concepts. By restricting the divisibility relation $|$ on the terms $\mathbb{T}_n$ to $|_{L,T}$, i. e. $t_1 \,|_{L,T}\, t_2$, if and only if $\epsilon(t_1) \,|_{L,\epsilon(T)}\, \epsilon(t_2)$ for t_1, t_2 in a finite subset $T \subset \mathbb{T}_n$, we obtain correspondingly the notions of *involutive reducibility* and *involutive normal forms.*

A finite set $F \subset \mathbb{W}_n$ is said to be *autoreduced*, if every $f \in F$ is in normal form modulo $F \setminus \{f\}$. It is *involutively autoreduced*, if no $f \in F$ contains a term $t = x^\mu \partial^\nu$ such that $\mathrm{le}\, f' \,|_{L,\mathrm{le}\,(F)}\, \epsilon(t)$ for some $f' \in F$. Note that it is crucial here to compute the multiplicative indices with respect to $\mathrm{le}\,(F)$. For this reason, we cannot simply say that f is in involutive normal form module $F \setminus \{f\}$.

Definition: Let $F = \{f_1, \dots, f_r\}$ be a weak involutive basis of the left ideal I. If there do not exist two indices $i \neq j$ such that $\mathrm{le}\,(f_i) \,|_{L,\mathrm{le}\,(F)}\, \mathrm{le}\,(f_j)$, then F is called a *(strong) involutive basis.*

One easily sees that the sum in (3) is direct, if and only if F is a strong involutive basis. Thus any such basis induces a Stanley decomposition of the given ideal into a direct sum of free modules over subalgebras of the Weyl algebra. In the case of commutative polynomials, this property allows for a trivial determination of the Hilbert function [Stanley, 1978].

THEOREM 4.1: *Let $I \subseteq \mathbb{W}_n$ be a left ideal. The finite set $F = \{f_1, \dots, f_r\} \subset I$ is an involutive basis of I for the multiplicative monomial order $\preceq$, if and only if every polynomial $f \in I$ possesses a unique involutive standard representation $f = \sum_{i=1}^r g_i f_i$ where $g_i \in \mathbb{K}[(\iota \circ N_{L,\mathrm{le}(F)} \circ \mathrm{le})(f_i)]$ and $\mathrm{lt}\,(g_i f_i) \preceq \mathrm{lt}\,(f)$. Any involutive basis F of I is also a Gröbner basis of I (for the order $\preceq$).*

Proof: We prove only the first statement; the second one is a trivial corollary.

One direction is simple. If any polynomial in the ideal I has a standard representation of the given form, then obviously $\langle F \rangle_L = I$ and F is at least a weak involutive basis. Because of the assumed uniqueness of the involutive standard representation, it is in fact a strong involutive basis.

For the converse, we first note that, by definition of a (weak) involutive basis, the set F generates I. Let f be an arbitrary element of I. By (3) it possesses a representation $f = \sum_{i=1}^r g_i f_i$ with $g_i \in \mathbb{K}[(\iota \circ N_{L,\mathrm{le}(F)} \circ \mathrm{le})(f_i)]$. As F is even a strong involutive basis, it is not possible that leading terms cancel in the addition. Thus $\mathrm{lt}\,(g_i f_i) \preceq \mathrm{lt}\,(f)$ for all i and only one generator f_j exists such that $\mathrm{lt}\,(g_j f_j) = \mathrm{lt}\,(f)$. By Prop. 3.1, this equation uniquely determines $\mathrm{lt}\,(g_j)$. Considering the polynomial $f - \mathrm{lt}\,(g_j) f_j \in I$ yields a similar equation for the next term in one of the coefficients g_i. Iteration uniquely determines all coefficients g_i of the involutive standard representation. $\square$

Note that the involutive standard representation is unique in contrast to the ordinary one with respect to an arbitrary Gröbner basis. This is a simple consequence of the fact that the sum in (3) is direct for involutive bases. A characteristic and defining property of Gröbner bases is that their leading terms generate the leading ideal. Similarly, involutive bases with respect to term orders are characterised by the fact that the leading terms involutively generate the leading ideal.

THEOREM 4.2: *Let $I \subseteq \mathbb{W}_n$ be a left ideal, $\preceq$ a term order on $\mathbb{T}_n$, and $F \subset I$ a finite set. If $\mathrm{le}\,(F)$ is a (weak) involutive basis of $\mathrm{le}\,(I)$ with respect to the division L, then F is a (weak) involutive basis of I. If F is a strong involutive basis, the converse is true, too.*

Proof: We may take without any modifications the proof given in [Seiler, 2001, Theorem 2.3.11] for the polynomial case. It is based on computing step by step an involutive normal form. The last statement was already shown in the proof of Theorem 4.1. $\square$

An obvious question is how we can explicitly compute an involutive basis for a given involutive division and a given order. For a term order, the simplest approach is the following one: we first compute a Gröbner basis, then we complete the leading terms of its elements to an involutive basis of the leading ideal. By Theorem 4.2 this yields the desired result. In fact, this algorithm was proposed by Sturmfels and White [1991] for the determination of Stanley decompositions in the case of commutative polynomials.

Computational experiments by Gerdt et al. [2001] have shown that in the polynomial case it is often better to use an extension of the "monomial" completion algorithm of Fig. 2. It is shown in Fig. 3 and uses two simple subalgorithms the details of which we omit, as they are obvious: $\texttt{InvAutoReduce}_{L,\preceq}(F)$ involutively autoreduces the given finite set F; $\texttt{InvNormalForm}_{L,\preceq}(g, F)$ computes the involutive normal form of g with respect to the set F. The algorithm automatically determines *strong* involutive bases because of the performed involutive autoreductions. In the shown form it can be directly applied to the computation of involutive bases with respect to term orders.

Algorithm: *Involutive Completion in* $\mathbb{W}_n$
Input: Finite subset $F \subset \mathbb{W}_n$, involutive division L on $\mathbb{N}_0^{2n}$, term order $\preceq$ on $\mathbb{T}_n$
Output: Strong involutive basis of the left ideal $\langle F \rangle$

$F \leftarrow \texttt{InvAutoReduce}_{L,\preceq}(F)$
`repeat`
  $\mathcal{S} \leftarrow \bigl\{\iota(j) \cdot f \;:\; f \in F,\, j \in \bar{N}_{L,\mathrm{le}\,(F)}(\mathrm{le}\,(f))\bigr\}$
  `repeat`
    $\bar{g} \leftarrow \min_{\preceq} \mathcal{S}$
    $\mathcal{S} \leftarrow \mathcal{S} \setminus \{\bar{g}\}$
    $g \leftarrow \texttt{InvNormalForm}_{L,\preceq}(\bar{g}, F)$
  `until` $(g \neq 0)$ `or` $(\mathcal{S} = \emptyset)$
  `if` $g \neq 0$ `then` $F \leftarrow \texttt{InvAutoReduce}_{L,\preceq}(F \cup \{g\})$
  `else return` F `end_if`
`until` $FALSE$

Figure 3: Completion for left ideals in $\mathbb{W}_n$

Theorem 4.3: *Let L be a continuous and constructive involutive division, $\preceq$ a term order on $\mathbb{T}_n$ and $F \subset \mathbb{W}_n$ a finite set such that the monoid ideal* $\mathrm{le}\,(\langle F \rangle)$ *possesses an involutive basis with respect to L. Then the algorithm in Fig. 3 terminates with an involutive basis of $\langle F \rangle$.*

Proof: As already mentioned in the introduction, the proof for the polynomial case (see e. g. [Gerdt and Blinkov, 1998a] or [Seiler, 2001]) can be used with some trivial modifications. □

It is a well-known property of involutive bases that, in contrast to Gröbner bases, their existence cannot be proven via the termination of a completion algorithm; in Theorem 4.3 we must require the existence of an involutive basis of the leading ideal. Of course, this condition is trivial for Noetherian divisions like the Janet division for which any monoid ideal has an involutive basis.

Obviously, all notions and results presented in this section can be trivially extended to the homogenised Weyl algebra $\mathbb{W}_n^{(h)}$. This requires the use of involutive divisions on $\mathbb{N}_0^{2n+1}$ and orders defined on $\mathbb{T}_n^{(h)}$. In particular, the algorithm presented in Fig. 3 remains valid without modifications.

It should be noted that Fig. 3 gives only the basic form of the involutive completion algorithm. Like for Buchberger's algorithm, a number of optimisations have been found (their effect is less profound, as one can show that many of the optimisations found for Buchberger's algorithm are automatically included in the involutive strategy). In the examples in Sect. 6 we will always use a more sophisticated form of the algorithm given by Gerdt and Blinkov [1998b], as it typically leads to smaller bases. This concerns in particular the Janet division which we will mainly use.

Remark: An involutive basis, in the polynomial case, has the nice property that the Hilbert polynomial of the ideal it generates can be read off immediately, as it defines a Stanley decomposition [Seiler, 2001, Sect. 2.5]. This can easily be transferred to involutive bases of left ideals in $\mathbb{W}_n$ as follows: consider a left ideal $I \subset \mathbb{W}_n$ having the strong involutive basis $F = \{f_1, \dots, f_r\}$ with respect to a degree compatible term order $\preceq$. We endow $\mathbb{W}_n$ with the Bernstein filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$, where the $\mathbb{K}$-vector spaces $\mathcal{F}_k$ are spanned by the terms $x^\alpha \partial^\beta$ with $\alpha + \beta \leq k$, respectively (see [Coutinho, 1995] for details). Then, denoting by σ_k $(k \in \mathbb{N})$ the canonical projection of $\mathcal{F}_k$ onto $\mathcal{F}_k/\mathcal{F}_{k-1}$

$$\left\{\sigma_{|\operatorname{le}(f_1)|}(f_1), \dots, \sigma_{|\operatorname{le}(f_r)|}(f_r)\right\}$$

is a strong involutive basis of the homogeneous graded ideal

$$\operatorname{gr}^{\mathcal{F}}(I) = \bigoplus_{k \geq 1} (I \cap \mathcal{F}_k)/\mathcal{F}_{k-1}$$

of the associated graded algebra of $\mathbb{W}_n$ with respect to $\mathcal{F}$ (which is just a commutative polynomial ring in $2n$ variables). If we define the Hilbert series of I as that of $\operatorname{gr}^{\mathcal{F}}(I)$, then, following Stanley [1978], we obtain

$$\mathcal{H}_I(\lambda) = \sum_{i=1}^{r} \frac{\lambda^{|\operatorname{le}(f_i)|}}{(1-\lambda)^{|N_{L,\operatorname{le}(F)}(f_i)|}} \,.$$

The Hilbert function and polynomial can be readily computed from $\mathcal{H}_I$.

5. Extension to Non-Term Orders I

In the last section we could prove the termination of our completion algorithm only for term orders. As a first step towards its extension to non-term orders, we examine the problem of lifting an involutive division defined on $\mathbb{N}_0^{2n}$ to a division defined on $\mathbb{N}_0^{2n+1}$. We simply write μ for the multi index $(\tilde{\mu}^1, \dots, \tilde{\mu}^{2n}) \in \mathbb{N}^{2n}$ obtained by dropping the entry $\tilde{\mu}^0$ (the degree of the homogenisation variable h) in the multi index $\tilde{\mu} = (\tilde{\mu}^0, \dots, \tilde{\mu}^{2n}) \in \mathbb{N}_0^{2n+1}$. Similarly, a set $\tilde{\mathcal{N}} \subset \mathbb{N}_0^{2n+1}$ yields a set $\mathcal{N} \subset \mathbb{N}_0^{2n}$ by projecting each element onto the last $2n$ entries.

Proposition 5.1: *Let L be an involutive division on $\mathbb{N}_0^{2n}$. For any finite set $\tilde{\mathcal{N}} \subset \mathbb{N}_0^{2n+1}$ and every $\tilde{\mu} \in \tilde{\mathcal{N}}$, we define $N_{L_h,\tilde{\mathcal{N}}}(\tilde{\mu})$ by:*

- *$0 \in N_{L_h,\tilde{\mathcal{N}}}(\tilde{\mu})$, if and only if $\tilde{\mu}^0 = \max_{\tilde{\nu}\in\tilde{\mathcal{N}}}\{\tilde{\nu}^0\}$,*
- *$0 < i \in N_{L_h,\tilde{\mathcal{N}}}(\tilde{\mu})$, if and only if $i \in N_{L,\mathcal{N}}(\mu)$.*

This determines an involutive division L_h on $\mathbb{N}_0^{2n+1}$.

Proof: Both conditions for an involutive division are easily verified. For the first one, choose $\tilde{\rho} \in C_{L_h,\tilde{\mathcal{N}}}(\tilde{\mu}) \cap C_{L_h,\tilde{\mathcal{N}}}(\tilde{\nu})$, where $\tilde{\mu}, \tilde{\nu} \in \tilde{\mathcal{N}}$. If $\tilde{\rho}^0 = \tilde{\mu}^0 = \tilde{\nu}^0$, the zero index can be ignored, and the properties of the involutive division L yield the desired result. If $\tilde{\rho}^0 = \tilde{\mu}^0 > \tilde{\nu}^0$, the index 0 must be multiplicative for $\tilde{\nu}$ requiring that the entry $\tilde{\nu}^0$ must be maximal among all elements of $\tilde{\mathcal{N}}$ and one obtains the contradiction $\tilde{\nu}^0 \geq \tilde{\mu}^0$. If $\tilde{\rho}^0$ is greater than both $\tilde{\mu}^0$ and $\tilde{\nu}^0$, the index 0 must be multiplicative for both requiring $\tilde{\mu}^0 = \tilde{\nu}^0$, so the index for h can be dropped and the properties of L invoked.

For the second condition we note that whether a multiplicative index $i > 0$ becomes non-multiplicative after adding elements to $\tilde{\mathcal{N}}$ is independent of the zero index and only determined by the involutive division L. If the maximal value of the index for h in $\tilde{\mathcal{N}}$ changes, 0 can only become non-multiplicative for elements in $\tilde{\mathcal{N}}$. □

Proposition 5.2: *If L is a Noetherian division, the same holds for L_h.*

Proof: Let $\tilde{\mathcal{N}} \subset \mathbb{N}_0^{2n+1}$ be an arbitrary finite subset. In order to show that there exists an L_h-completion of $\tilde{\mathcal{N}}$, we first determine a finite L-completion $\hat{\mathcal{N}} \subset \mathbb{N}_0^{2n}$ of $\mathcal{N}$ which always exists, as by assumption L is Noetherian. Next, we define a finite subset $\tilde{\mathcal{G}} \subset \langle \tilde{\mathcal{N}} \rangle$:

$$\tilde{\mathcal{G}} = \left\{ \tilde{\mu} \in \mathbb{N}_0^{2n+1} \;:\; \mu \in \hat{\mathcal{N}} \text{ and } \tilde{\mu}^0 \leq \max_{\tilde{\eta}\in\tilde{\mathcal{N}}} \tilde{\eta}^0 \right\} \cap \langle \tilde{\mathcal{N}} \rangle .$$

We claim that $\tilde{\mathcal{G}}$ is an L_h-completion of $\tilde{\mathcal{N}}$. By construction, we have both $\tilde{\mathcal{G}} \subset \langle \tilde{\mathcal{N}} \rangle$ and $\tilde{\mathcal{N}} \subseteq \tilde{\mathcal{G}}$, i. e. we are left to show that $\tilde{\mathcal{G}}$ is involutive.

So let $\tilde{\mu} \in \langle \tilde{\mathcal{G}} \rangle$ be arbitrary. Then by construction of $\tilde{\mathcal{G}}$ we can find $\tilde{\nu} \in \tilde{\mathcal{G}}$ with $\nu \,|_{L,\hat{\mathcal{N}}}\, \mu$. Moreover, the definition of $\tilde{\mathcal{G}}$ guarantees that we can choose $\tilde{\nu}$ in such a way that either $\tilde{\nu}^0 = \tilde{\mu}^0$ or $\tilde{\nu}^0 = \max_{\tilde{\eta}\in\tilde{\mathcal{G}}} \tilde{\eta}^0 < \tilde{\mu}^0$ holds. In the former case we trivially have $\tilde{\nu}\,|_{L_h,\tilde{\mathcal{G}}}\,\tilde{\mu}$; in the latter case we have $0 \in \mathcal{N}_{L_h,\tilde{\mathcal{G}}}(\tilde{\nu})$ (see Prop. 5.1). So again we obtain $\tilde{\mu} \in \langle\, \tilde{\mathcal{G}}\,\rangle_{L_h}$ as required. □

Proposition 5.3: *If L is a continuous division, the same holds for L_h.*

Proof: Let $(\tilde{\mu}_i)_{i=1,\dots,r}$ with $\tilde{\mu}_i \in \tilde{\mathcal{N}}$ be a sequence as described in the definition of continuity. First we observe that the sequence $(\tilde{\mu}_i^0)$ is monotonically increasing. If $\tilde{\mu}_i^0$ is not maximal among the entries $\tilde{\rho}^0$ for $\tilde{\rho} \in \tilde{\mathcal{N}}$, no multiplicative divisor of $\tilde{\mu}_i + 1_j$ in $\tilde{\mathcal{N}}$ can have a smaller entry for h: if $\tilde{\mu}_i^0$ is maximal, the index 0 cannot be non-multiplicative for $\tilde{\mu}_i$ and any involutive divisor in $\tilde{\mathcal{N}}$ must also be maximal in the zero entry. Thus it suffices to look at those parts of the sequence where equality in the zero entries holds. But there the inequality of the $\tilde{\mu}_i$ follows from the continuity of the underlying division L. □

Unfortunately, the constructivity of the lifted division for a constructive L is much harder to show. We present here the result only for globally defined divisions and the Janet division. The proof imitates the one of the constructivity of the Janet division given by Gerdt and Blinkov [1998a].

Proposition 5.4: *Let L be an involutive division on $\mathbb{N}_0^{2n}$. If L is either globally defined or the Janet division, then the lifted division L_h is constructive.*

Proof: We select a finite set $\tilde{\mathcal{N}} \subset \mathbb{N}_0^{2n+1}$, a multi index $\tilde{\mu} \in \tilde{\mathcal{N}}$ and a non-multiplicative index i of $\tilde{\mu}$ such that the conditions in the definition of constructivity are fulfilled. Assume that there exists a $\tilde{\rho} \in \tilde{\mathcal{N}}$ such that $\tilde{\mu}+1_i = \tilde{\rho}+\tilde{\sigma}+\tilde{\tau}$ with $\tilde{\rho}+\tilde{\sigma} \in C_{L_h,\tilde{\mathcal{N}}}(\tilde{\rho})$ and $\tilde{\rho}+\tilde{\sigma}+\tilde{\tau} \in C_{L_h,\tilde{\mathcal{N}}\cup\{\tilde{\rho}+\tilde{\sigma}\}}(\tilde{\rho}+\tilde{\sigma})$. Let L be a globally defined division. If $i = 0$, $\tilde{\mu}^0 + 1 = \tilde{\rho}^0 + \tilde{\sigma}^0 + \tilde{\tau}^0$ implies that $\tilde{\sigma}^0 = \tilde{\tau}^0 = 0$: for $\tilde{\sigma}^0 > 0$, we would have (0 is multiplicative for $\tilde{\rho}$) $\tilde{\rho}^0 > \tilde{\mu}^0 \geq \tilde{\rho}^0 + \tilde{\sigma}^0 > \tilde{\rho}^0$ and for $\tilde{\sigma}^0 = 0$, $\tilde{\tau}^0 > 0$ a similar contradiction appears. If $i > 0$, the argumentation is simple. A global division is always constructive, as adding further elements to $\mathcal{N}$ does not change the multiplicative indices. But the same holds for the indices $k > 0$ in the lifted division L_h. Thus under the above conditions $\tilde{\mu}+1_i \in \langle\,\tilde{\mathcal{N}}\,\rangle_{L_h}$ contradicting the assumptions.

For L the Janet division, we construct a sequence $(\tilde{\rho}_k)$ as follows. We start with $\tilde{\rho}_1 = \tilde{\rho}$ and $\tilde{\phi}_1 = \tilde{\sigma} + \tilde{\tau}$. In each step, we select first a non-multiplicative index j_k for $\tilde{\rho}_k$ such that $\tilde{\phi}_k^{j_k} > 0$ and then a multiplicative divisor $\tilde{\rho}_{k+1}$ of $\tilde{\rho}_k + 1_{j_k}$:

$$\begin{aligned}\tilde{\mu} + 1_i &= \tilde{\rho}_k + \tilde{\phi}_k = \tilde{\rho}_k + 1_{j_k} + \tilde{\phi}_k - 1_{j_k}\\ &= \tilde{\rho}_{k+1} + \underbrace{\tilde{\sigma}_{k+1} + \tilde{\phi}_k - 1_{j_k}}_{=\tilde{\phi}_{k+1}}\end{aligned}$$

If in some step k the multi index $\tilde{\phi}_k$ consists only of multiplicative indices for $\tilde{\rho}_k$, we have obviously a contradiction. It remains to show that the replacement of

$\tilde{\rho}_k + 1_{j_k}$ is always possible. The second condition in the definition of constructivity requires for this that $\tilde{\rho}_k + 1_{j_k}$ is a proper divisor of $\tilde{\mu} + 1_{j_k}$. From considerations about the lengths of the involved multi indices, this is certainly the case if in each step $|\tilde{\phi}_k|$ is greater than one (note that $|\tilde{\sigma}|$ and $|\tilde{\tau}|$ cannot be zero). We prove the following inequalities for the *lexicographic* order $\succeq$ on $\mathbb{N}_0^{2n+1}$:

1. $\tilde{\rho}_1 = \tilde{\rho} \succ \tilde{\mu}$: Let s and t, respectively, be the position of the left-most non-vanishing entry in $\tilde{\sigma}$ and $\tilde{\tau}$. Assume first that $i = 0$. Then it follows from $\tilde{\mu}^0 + 1 = \tilde{\rho}^0 + \tilde{\sigma}^0 + \tilde{\tau}^0$ that for $s > 0$ we have $\tilde{\rho}^0 \leq \tilde{\mu}^0$ and that for $s = 0$, $t > 0$ we have $(\tilde{\rho} + \tilde{\sigma})^0 \leq \tilde{\mu}^0$, both leading to a contradiction. If $i = s = 0$, then $\tilde{\rho}^0 = \tilde{\mu}^0 + 1$ and thus $\tilde{\rho} \succ \tilde{\mu}$. For $i > 0$, we claim that $i < s$ and $i < t$, from which the statement follows directly. This can be obtained by a simple reasoning about the properties of the Janet division (completely analogously to the proof of Prop. 4.13 by Gerdt and Blinkov [1998a]).
2. $\tilde{\rho}_{k+1} \succ \tilde{\rho}_k$: We have $\tilde{\rho}_k + 1_{j_k} = \tilde{\rho}_{k+1} + \tilde{\sigma}_{k+1}$. If $j_k \neq 0$, this can be seen by projecting to $\mathbb{N}_0^{2n}$ and applying the definition of the Janet division. For $j_k = 0$, we first consider the case where $\tilde{\sigma}_{k+1}^0 \neq 0$. This means that 0 is multiplicative for $\tilde{\rho}_{k+1}$ but non-multiplicative for $\tilde{\rho}_k$, although $\tilde{\rho}_k^0 \geq \tilde{\rho}_{k+1}^0$ which is not possible. Thus $\tilde{\sigma}_{k+1}^0 = 0$, $\tilde{\rho}_{k+1}^0 > \tilde{\rho}_k^0$ and therefore $\tilde{\rho}_{k+1} \succ \tilde{\rho}_k$.

Now suppose we arrive at a situation where $\tilde{\mu} + 1_i = \tilde{\rho}_k + \tilde{\phi}_k$ with $\tilde{\phi}_k = 1_{j_k}$. We can assume without loss of generality that we have chosen $\tilde{\mu}$ maximal among all multi indices $\tilde{\nu}$ with $\tilde{\mu} + 1_i = \tilde{\nu} + 1_\ell$, ℓ multiplicative for $\tilde{\nu}$. By this we get the contradiction $\tilde{\mu} \succeq \tilde{\rho}_k \succ \cdots \succ \tilde{\rho}_1 \succ \tilde{\mu}$.

If the sequence $(\tilde{\rho}_i)$ constructed above is infinite, it must have elements occurring repeatedly, as $\tilde{\mu} + 1_i$ has only finitely many divisors. But this contradicts the continuity of the lifted Janet division. □

6. Extension to Non-Term Orders II

Based on our results in the last section, an extension of the algorithm in Fig. 3 to non-term orders is possible. We know that we can lift any multiplicative monomial order $\preceq$ on $\mathbb{T}_n$ to a term order $\preceq_h$ on $\mathbb{T}_n^{(h)}$. We also know that any involutive division L on $\mathbb{N}_0^{2n}$ induces a division L_h on $\mathbb{N}_0^{2n+1}$. So it is a natural thought to homogenise the input set F and then to determine with the above algorithm an involutive basis of $\langle F^{(h)} \rangle$ with respect to L_h and $\preceq_h$.

We have already shown that L_h is continuous, if L is, and that at least for globally defined divisions and the Janet division constructivity is preserved. It remains to show that the existence of an involutive basis is also preserved under the lifting.

Proposition 6.1: *If the left ideal $\langle F \rangle \subseteq \mathbb{W}_n$ possesses an involutive basis with respect to the Noetherian division L and the multiplicative monomial order $\preceq$, then the left ideal $\langle F^{(h)} \rangle \subseteq \mathbb{W}_n^{(h)}$ generated by the homogenisations of the elements in the basis F possesses an involutive basis with respect to the division L_h and the term order $\preceq_h$.*

Proof: From [Saito et al., 2000, Prop. 1.2.3] we know that $\langle F^{(h)} \rangle$ possesses a Gröbner basis $\tilde{G}$ with respect to $\preceq_h$. Using Prop. 5.2 we see that there is a finite L_h-completion $\hat{\mathcal{N}}$ of $\mathrm{le}^{(h)}_{\preceq_h}(\tilde{G})$. Moreover, as $\tilde{G}$ is a Gröbner basis of $\langle F^{(h)} \rangle$, the spans of $\mathrm{le}^{(h)}_{\preceq_h}(\tilde{G})$ and $\mathrm{le}^{(h)}_{\preceq_h}(\langle F^{(h)} \rangle)$ coincide, thus $\hat{\mathcal{N}}$ is an involutive basis of $\mathrm{le}^{(h)}_{\preceq_h}(\langle F^{(h)} \rangle)$ with respect to the division L_h. So from Theorem 4.2 we see that an involutive basis $\tilde{H}$ of $\langle F^{(h)} \rangle$ with respect to the division L_h is given by

$$\tilde{H} = \left\{ h^\lambda x^\mu \partial^\nu \cdot \tilde{g} \;:\; \tilde{g} \in \tilde{G} \;\wedge\; \mathrm{le}^{(h)}_{\preceq_h}(h^\lambda x^\mu \partial^\nu \cdot \tilde{g}) \in \hat{\mathcal{N}} \right\} .$$

This set is obviously finite. □

Hence our lifting leads to a situation where we can apply Theorem 4.3. Unfortunately, the dehomogenisation of the strong involutive basis computed in $\mathbb{W}_n^{(h)}$ does not necessarily lead to a *strong* involutive basis in $\mathbb{W}_n$. But one can show that one obtains a weak involutive basis which is also a Gröbner basis.

THEOREM 6.1: *Let $\tilde{H}$ be a strong involutive basis with respect to L_h and $\preceq_h$ for the left ideal $\langle F^{(h)} \rangle \subseteq \mathbb{W}_n^{(h)}$. Then the dehomogenisation H is both a weak involutive and a Gröbner basis for $\langle F \rangle \subseteq \mathbb{W}_n$ with respect to L and $\preceq$.*

Proof: An integer $a \geq 0$ exists for any $f \in \langle F \rangle$ such that $\tilde{f} = h^a f^{(h)} \in \langle F^{(h)} \rangle$. Setting $h = 1$ in the unique involutive standard representation of $\tilde{f}$ with respect to the involutive basis $\tilde{H}$ yields a representation of f with respect to the basis $H = \{f_1, \dots, f_r\}$ of the form $f = \sum_{i=1}^r g_i f_i$ with $g_i \in \mathbb{K}[(\iota \circ N_{L,\mathrm{le}(H)} \circ \mathrm{le})(f_i)]$. Thus H is a weak involutive basis of the ideal $\langle F \rangle$ for the division L. As the set $\tilde{H}$ is a strong involutive basis of $\langle F^{(h)} \rangle$ with respect to a term order, it is by Theorem 4.1 simultaneously a Gröbner basis of $\langle F^{(h)} \rangle$. Thus H is a Gröbner basis of the ideal $\langle F \rangle$ by Theorem 3.1. □

Example: An important class of left ideals in the Weyl algebra $\mathbb{W}_n$ are the Gelfand-Kapranov-Zelevinsky (GKZ) systems [Gelfand et al., 1989] describing hypergeometric functions. Any such system is described by a $d \times n$ integer matrix A of rank d and a vector $\beta \in \mathbb{K}^d$. Its generators are

$$f_i = \sum_{j=1}^n A_{ij} x_j \partial_j - \beta_i \,, \qquad \text{for } i = 1, \dots, d$$

and in addition for every pair of vectors $u, v \in \mathbb{N}_0^n$ with $Au = Av$ the generator $f_{uv} = \partial^u - \partial^v$. The generators f_i express certain homogeneity relations of the solutions of the corresponding linear system of partial differential equations; the operators f_{uv} generate the toric ideal associated with A (Sturmfels [1996] gives concrete algorithms for their construction).

If we take as a concrete example

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} , \qquad \beta = \begin{pmatrix} -1 \\ 1 \end{pmatrix} ,$$

then the corresponding GKZ system is generated by $2x\partial_x+y\partial_y+1$, $2z\partial_z+y\partial_y-1$ and $\partial_x\partial_z-\partial_y^2$. The solution space of the induced linear system of partial differential equations is spanned by the two familiar expressions $(-y\pm\sqrt{y^2-4xz})/2x$, i. e. the solutions of the quadratic equation $x\lambda^2+y\lambda+z=0$.

The construction of formal series solutions of such hypergeometric systems requires Gröbner bases with respect to non-term orders defined by weight vectors of the form $(-\xi,\xi)\in\mathbb{R}^{2n}$ [Saito et al., 2000]. We take $\xi=(1,0,0)$ and refine by the degree reverse lexicographic term order with $y\prec x\prec z\prec\partial_y\prec\partial_x\prec\partial_z$ (note the permutation of x and y). As involutive division we use the Janet division with the permutation (6 4 5 3 1 2) corresponding to the permutation taken in the order. The involutive completion algorithm yields then the following basis with eight generators:

$$
\begin{aligned}
H=\Big\{\ &\underline{2x\partial_x}+y\partial_y+1,\ \underline{2z\partial_z}+y\partial_y-1,\ \underline{\partial_x\partial_z}-\partial_y^2,\\
&\underline{y\partial_y\partial_z}+\partial_z+2x\partial_y^2,\ \underline{2z\partial_y\partial_z}+y\partial_y^2,\ \underline{y\partial_x\partial_y}-\partial_x+2z\partial_y^2,\\
&\underline{2x\partial_x\partial_y}+y\partial_y^2+2\partial_y,\ \underline{y^2\partial_y^2}+y\partial_y-1-4xz\partial_y^2\ \Big\}.
\end{aligned}
$$

As one can see from the underlined leading terms, we have actually obtained a strong Janet basis. In this case, a reduced Gröbner basis exists and consists of six generators, namely one can drop the fifth and the seventh generator in the basis above which are just ∂_y times the first and the second generator, respectively.

It follows from the definition of the lifted term order $\preceq_h$ that the representation $f=\sum_{i=1}^r g_if_i$ constructed in the proof of Theorem 6.1 is a standard representation with respect to the Gröbner basis H, i. e. $\operatorname{le}(g_if_i)\preceq\operatorname{le}(f)$. However, in general it is not unique and thus H is not necessarily a strong involutive basis.

Example: Consider the left ideal generated by $F=\{\underline{1}+x_1+x_2,\underline{\partial_2}-\partial_1\}$. We take the multiplicative monomial order induced by the weight vector $(-1,-1,1,1)$ and refined by a term order for which $\partial_2\succ\partial_1\succ x_2\succ x_1$. Then the underlined terms are the leading ones. One easily checks that F is a Gröbner basis for this order. If we apply the Pommaret division together with the permutation (4 3 2 1), then all variables are multiplicative for each generator and F is a weak Pommaret basis, too.

Obviously, F is neither a reduced Gröbner basis nor a strong Pommaret basis, as 1 is a (multiplicative) divisor of ∂_2. However, it is easy to see that the left ideal $I=\langle F\rangle$ does not possess a reduced Gröbner basis or a strong Pommaret basis. Indeed, we have $\operatorname{le}(I)=\mathbb{N}_0^{2n}$ and thus such a basis had to consist of only a single generator; but I is not a principal ideal.

For the Janet division J the situation is more favourable. It follows immediately from its definition that any weak Janet basis in which no two generators possess the same leading exponent is a strong basis. Two simple modifications of the involutive completion algorithm achieve that the dehomogenisation H always has this property.

As a first modification we perform during the completion only head reductions, i. e. in normal form computations and autoreductions we always treat only the leading term. Obviously, this does not affect the correctness or the termination of the algorithm, as reductions to zero remain unchanged.

Assume that at some intermediate stage of the completion the basis $\tilde{H}$ contains two polynomials $\tilde{f}$, $\tilde{g}$ such that $\mathrm{le}\,(\tilde{g}) = \mathrm{le}\,(\tilde{f}) + 1_0$. If $\tilde{g} = h\tilde{f}$, we have $f = g$ after the dehomogenisation and no obstruction to a strong basis appears. Otherwise we note that, by definition of the lifted division J_h, the homogenisation variable h is non-multiplicative for $\tilde{f}$. Thus at some later stage the algorithm must consider the non-multiplicative product $h\tilde{f}$ (if it was already treated, $\tilde{H}$ would not be involutively head autoreduced).

In the usual algorithm, we then determine the involutive normal form of the polynomial $h\tilde{f}$; the first step of this computation is to replace $h\tilde{f}$ by $h\tilde{f} - \tilde{g}$. Alternatively, we may proceed instead as follows. The polynomial $\tilde{g}$ is removed from the basis $\tilde{H}$ and replaced by $h\tilde{f}$. Then we continue by determining the involutive normal form of $\tilde{g}$ with respect to the new basis. Note that this modification concerns only the situation that a multiplication by h has been performed and that the basis $\tilde{H}$ contains already an element with the same leading exponent as the obtained polynomial.

If the final output $\tilde{H}$ of the thus modified completion algorithm contains two polynomials $\tilde{f}$ and $\tilde{g}$ such that $\mathrm{le}\,(\tilde{g})$ and $\mathrm{le}\,(\tilde{f})$ differ only in the zeroth entry, then either $\tilde{g} = h^a\tilde{f}$ or $\tilde{f} = h^a\tilde{g}$ for some $a \in \mathbb{N}$. Thus the dehomogenisation yields a basis H where all elements possess different leading terms. This implies that H is a strong Janet basis. Again this modification does not affect the correctness and the termination of the algorithm. As the Janet division is Noetherian, these considerations yield together with Prop. 5.2 the following result.

Theorem 6.2: *Let $\preceq$ be an arbitrary multiplicative monomial order. Then every left ideal $I \subseteq \mathbb{W}_n$ possesses a strong Janet basis for $\preceq$.*

Note that the second modification only achieves its goal, if we really restrict to head reductions. Otherwise it is possible that some terms other than the leading term in $h\tilde{f}$ are reducible but not the corresponding terms in $\tilde{f}$. In this case we could still find after the dehomogenisation two different generators with the same leading term.

Example: As a simple example we consider in $\mathbb{W}_3$ the left ideal generated by the set $F = \{\partial_z - y\partial_x,\, \partial_y\}$. If we apply the usual involutive completion algorithm, we obtain for the weight vector $(-1, 0, 0, 1, 0, 0)$ refined by the degree reverse lexicographic order and the Janet division with the permutation (6 5 4 3 2 1) the following weak basis with nine generators:

$$H_1 = \left\{\; \partial_x,\; \partial_y,\; \partial_z,\; \partial_x\partial_z,\; \partial_y\partial_z,\; y\partial_x,\; \underline{y\partial_x} + \partial_z,\; y\partial_x\partial_z,\; \underline{y\partial_x\partial_z} + \partial_z^2 \;\right\} .$$

As one can see from the last four generators, it is not a strong basis. Applying the modified algorithm yields the following basis with seven generators:

$$H_2 = \left\{ \underline{\partial_x} + \partial_y\partial_z,\ \partial_y,\ \partial_z,\ \partial_x\partial_z,\ \partial_y\partial_z,\ \underline{y\partial_x} + \partial_z,\ \underline{y\partial_x\partial_z} + \partial_z^2 \right\} .$$

Obviously, we now have a strong basis, as all leading terms are different.

This example also demonstrates the profound effect of the homogenisation. A strong Janet or Pommaret basis of $\langle F \rangle$ is simply given by $H = \{\partial_x, \partial_y, \partial_z\}$ which is simultaneously a reduced Gröbner basis. In $\langle F^{(h)} \rangle$ many reductions are not possible because the terms contain different powers of h. However, this is a general problem of all approaches to Gröbner bases for multiplicative monomial orders and not specific for the involutive approach.

In this particular case, one could have applied the involutive completion algorithm directly to the original set F and it would have terminated with the minimal basis H, although we are using a non-term order. Unfortunately, it is not clear how to predict when infinite reduction chains appear in normal form computations for non-term orders, so that one does not know in advance whether one may dispense with the homogenisation.

7. Conclusions

We showed that involutive bases can be defined and constructed for left ideals in the Weyl algebra with respect to both term and non-term orders. The latter ones are, however, rather difficult to handle, as normal form computations do not necessarily terminate and normal form arguments are even more important for involutive bases than for Gröbner bases. As not every ideal possesses a *reduced* Gröbner basis for a non-term order (see the last but one example), it is not surprising that *strong* involutive bases do not always exist.

But we could show that we can always find a weak involutive basis which is simultaneously a Gröbner basis. Thus one of the most important applications of involutive divisions, namely to provide an alternative approach to the construction of Gröbner bases, is possible even for non-term orders. Furthermore, we could show that one can always construct a strong Janet basis.

Constructivity is unfortunately a very technical concept. An open question is whether the lift of any constructive division to the homogenised Weyl algebra yields again a constructive division. From a practical point of view, the Janet and the Pommaret division are surely the most important involutive divisions and for them we could prove that the lift preserves all relevant properties.

We did not discuss complexity and efficiency issues. As one can see in the proof of Prop. 6.1, the lifted involutive basis will generally contain many elements that become identical upon dehomogenisation. But the same happens in the lift of Buchberger's algorithm. A careful treatment of these elements is crucial for an efficient implementation of either approach.

Within the computer algebra system *MuPAD*[¶] we implemented the involutive completion algorithm (based on the optimised form proposed by Gerdt and Blinkov [1998b]) for ideals in arbitrary polynomial algebras of solvable type (in the generalised sense of [Seiler, 2001]). For the special case of the Weyl algebra, this implementation provides tools for the (de)homogenisation of polynomials and for the lift of both multiplicative monomial orders and involutive divisions. However, no special treatment of the above mentioned problem has been designed. Furthermore, the modified algorithm for the construction of strong Janet bases has been implemented. All examples in this article have been computed with the help of our package.

References

J. Apel. A Gröbner approach to involutive bases. *J. Symb. Comp.*, 19:441–457, 1995.

J. Apel. Theory of involutive divisions and an application to Hilbert function computations. *J. Symb. Comp.*, 25:683–704, 1998.

A. Assi, F.J. Castro-Jiménez, and J.M. Granger. How to calculate the slopes of a $\mathcal{D}$-module. *Compos. Math.*, 104:107–123, 1996.

J. Briançon and P. Maisonobe. Idéaux de germes d'opérateurs différentiels à une variable. *Enseign. Math.*, 30:7–38, 1984.

J. Calmet, M. Hausdorf, and W.M. Seiler. A constructive introduction to involution. In R. Akerkar, editor, *Proc. Int. Symp. Applications of Computer Algebra — ISACA 2000*, pages 33–50. Allied Publishers, New Delhi, 2001.

F.J. Castro-Jiménez. *Théorème de Division pour les Opérateurs Différentiels et Calcul des Multiplicités.* PhD thesis, Université Paris VII, 1984.

F.J. Castro-Jiménez. Calculs effectifs pour les idéaux d'opérateurs différentiels. *Travaux en Cours*, 24:1–19, 1987.

F.J. Castro-Jiménez and L. Narváez-Macarro. Homogenising differential operators. Preprint Facultad de Matemáticas, Univ. Sevilla, 1997.

S.C. Coutinho. *A Primer of Algebraic $\mathcal{D}$-Modules.* London Mathematical Society Student Texts 33. Cambridge University Press, Cambridge, 1995.

A. Galligo. Some algorithmic questions on ideals of differential operators. In B.F. Caviness, editor, *Proc. EUROCAL '85, vol. 2*, Lecture Notes in Computer Science 204, pages 413–421. Springer-Verlag, Berlin, 1985.

K. Gatermann. *Computer Algebra Methods for Equivariant Dynamical Systems.* Lecture Notes in Mathematics 1728. Springer, Berlin, 2000.

[¶]See `www.mupad.de` for more information.

I.M. Gelfand, A.V. Zelevinsky, and M.M. Kapranov. Hypergeometric functions and toral manifolds. *Funct. Anal. Appl.*, 23:94–106, 1989.

V.P. Gerdt. Completion of linear differential systems to involution. In V.G. Ghanza, E.W. Mayr, and E.V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing — CASC '99*, pages 115–137. Springer, Berlin, 1999.

V.P. Gerdt and Yu.A. Blinkov. Involutive bases of polynomial ideals. *Math. Comp. Simul.*, 45:519–542, 1998a.

V.P. Gerdt and Yu.A. Blinkov. Minimal involutive bases. *Math. Comp. Simul.*, 45: 543–560, 1998b.

V.P. Gerdt, Yu.A. Blinkov, and D.A. Yanovich. Construction of Janet bases II: Polynomial bases. In V.G. Ghanza, E.W. Mayr, and E.V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing — CASC 2001*, pages 249–263. Springer-Verlag, Berlin, 2001.

E.L. Ince. *Ordinary Differential Equations.* Dover, New York, 1956.

M. Janet. *Leçons sur les Systèmes d'Équations aux Dérivées Partielles.* Cahiers Scientifiques, Fascicule IV. Gauthier-Villars, Paris, 1929.

D. Lazard. Gröbner bases, Gaussian elimination and resolution of systems of algebraic equations. In J.A. Hulzen, editor, *Proc. EUROCAL*, Lecture Notes in Computer Science 162, pages 146–156. Springer-Verlag, 1983.

T. Oaku. Algorithms for the b-function and $\mathcal{D}$-modules associated with a polynomial. *J. Pure Appl. Alg.*, 117&118:495–518, 1997.

T. Oaku, N. Takayama, and U. Walther. A localization algorithm for $\mathcal{D}$-modules. *J. Symb. Comp.*, 29:721–728, 2000.

C. Riquier. *Les Systèmes d'Équations aux Derivées Partielles.* Gauthier-Villars, Paris, 1910.

M. Saito, B. Sturmfels, and N. Takayama. *Gröbner Deformations of Hypergeometric Differential Equations.* Algorithms and Computation in Mathematics 6. Springer-Verlag, Berlin, 2000.

W.M. Seiler. Involution — the formal theory of differential equations and its applications in computer algebra and numerical analysis. Habilitation thesis, Dept. of Mathematics, Universität Mannheim, Germany, 2001.

R.P. Stanley. Hilbert functions of graded algebras. *Adv. Math.*, 28:57–83, 1978.

B. Sturmfels. *Gröbner Bases and Convex Polytopes.* University Lecture Series 8. AMS, Providence, 1996.

B. Sturmfels and N. White. Computing combinatorial decompositions of rings. *Combinatorica*, 11:275–293, 1991.

A.Yu. Zharkov and Yu.A. Blinkov. Involution approach to solving systems of algebraic equations. In G. Jacob, N.E. Oussous, and S. Steinberg, editors, *Proc. Int. IMACS Symp. Symbolic Computation*, pages 11–17, Lille, 1993.